Advanced Engineering Mathematics	**Homework Two**

Introduction to Linear Vectors Spaces : Span, Linear Independence, Basis, Dimension

Text: 7.1-7.3, 7.5, 7.7-7.8 — Lecture Slides: 5-6

Quote of Homework Two

Don Juan Matus: The answer is very simple. He must not run away. He must defy his fear, and in spite of it he must take the next step in learning, and the next, and the next. He must be fully afraid, and yet he must not stop. That is the rule!

Carlos Castaneda - The Teachings of Don Juan: A Yaqui Way of Knowledge (1968)

1. Vocabulary of Vector Spaces

Given,

$$\mathbf{A}_1 = \begin{bmatrix} 5 & 3 \\ -4 & 7 \\ 9 & -2 \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} 22 \\ 20 \\ 15 \end{bmatrix},$$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -5 \\ 7 \\ 8 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ h \end{bmatrix},$$

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} -3 \\ 9 \\ -6 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} 5 \\ -7 \\ h \end{bmatrix},$$

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix},$$

$$\mathbf{A}_2 = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}.$$

1.1. **Linear Combinations.** Is $\mathbf{b}_1$ a linear combination of the columns of $\mathbf{A}_1$?

1.2. **Linear Dependence.** Determine all values for h such that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ forms a linearly dependent set.

1.3. **Linear Independence.** Determine all values for h such that $S = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ forms a linearly independent set.

1.4. **Spanning Sets.** How many vectors are in $S = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$? How many vectors are in span(S)? Is $\mathbf{y} \in$span(S)?

1.5. **Matrix Spaces.** Is $\mathbf{b}_2 \in \text{Nul}(\mathbf{A}_2)$? Is $\mathbf{b}_2 \in \text{Col}(\mathbf{A}_2)$?

2. The Spaces Defined by Linear Transformations

Given,

$$\mathbf{A} = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \end{bmatrix}.$$

2.1. **Null Space.** Determine a basis and the dimension of Nul$(\mathbf{A})$.

2.2. **Column Space.** Determine a basis and the dimension of Col$(\mathbf{A})$.

2.3. **Row Space.** Determine a basis and the dimension of Row $\mathbf{A}$. What is the Rank of $\mathbf{A}$?

3. Abstract Vector Spaces - Function Spaces

Given,

(1)
$$\left[m\frac{d^2}{dt^2} + k\right] y = 0,\ \ m, k \in \mathbb{R}^+$$

(2)
$$\left[\frac{d}{dt} - \mathbf{A}\right] \mathbf{Y} = 0,\ \ \mathbf{A} \in \mathbb{R}^{2\times 2}$$

3.1. **Equivalence of Equations.** Find the change of variables that maps (1) onto (2) and using this define **Y** and **A**.

3.2. **Function Spaces.** Find the general solution to (2) and for $m = k = 1$ sketch its associated real phase-portrait.

4. Introduction to Infinite Dimensional Spaces

Given,

(3)
$$y'' + \lambda y = 0,\ \ \lambda \in \mathbb{R}$$

(4)
$$y(0) = 0,\ \ y(\pi) = 0.$$

4.1. **General Solution.** Find the general solution to the ODE (3).

4.2. ***Boundary Value Problem.*** Show that the nontrivial solutions to (3) that satisfy the associated boundary conditions (4) requires that $\lambda = n^2 \in \mathbb{N}$.

4.3. **Infinite Dimensional Space.** Show that $u_n(x,t) = \sin(nx)e^{-n^2 t}$ satisfies $[\partial_t - \partial_{xx}]\, u = 0$.

5. *Hilbert Spaces*

Given,

(5)
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

5.1. **General Solutions to the *Heat Equation.*** Justify that $u(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx) e^{-n^2 t}$ is a solution to (5).

5.2. **Abstract Inner Products.** Show that $\langle f, g\rangle = (f, g) = \int_{-\pi}^{\pi} f(x)g(x)\, dx$ satisfies the three axioms of a *Real Inner Product Space* found on page 326.

5.3. **Orthogonality Relation.** Show that $\langle \sin(nx), \sin(mx)\rangle = \pi\delta_{nm}$ for all $n, m \in \mathbb{N}$.

5.4. **Initial Conditions and Unknown Constants.** Suppose the solution to (5) is initially known to be $u(x,0) = f(x)$. Using the previous orthogonality relations, show that $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\, dx$.