Spherical waves

Plane waves, being infinite in extent, are obviously kind of an idealization or a limiting case. Another option is the spherical wave: a solution to the wave equation and Maxwell's equations that has some kind of $e^{i(kr-\omega t)}$ dependence, indicating propagation radially outward (like what you might expect a point source to produce).

It's actually pretty hard to derive the form of spherical waves directly without using certain approximations (see, for example, something called the paraxial wave equation). Fortunately, we can guess at the form of the solution and do some construction.

The solution ought to have a propagating term like $e^{i(kr-\omega t)}$, and the overall amplitude of either E or B ought to go like $1/r$ so that the intensity falls off as $1/r^2$.

So let's guess:
$$\vec{E}(r,t) = \frac{E_0}{r} e^{i(kr-\omega t)} \hat{\theta}$$

And let's see if we can make it satisfy Maxwell's equations.

Start with Faraday's law to get the form of B:
$$\nabla \times \vec{E} = \frac{\hat{\phi}}{r} \frac{\partial}{\partial r}(r E_\theta) = \frac{\hat{\phi}}{r}\frac{\partial}{\partial r}\left(E_0 e^{i(kr-\omega t)}\right) = \frac{ikE_0}{r} e^{i(kr-\omega t)} \hat{\phi} = -\partial \vec{B}/\partial t$$

$$\Rightarrow \quad \vec{B} = \frac{k}{\omega}\frac{E_0}{r} e^{i(kr-\omega t)} \hat{\phi}$$

So far so good. That looks like a reasonable B. But now things start to fall apart. Check $\nabla \cdot \vec{E} = 0$:

$$\nabla \cdot \vec{E} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta \frac{E_0}{r} e^{i(kr-\omega t)}\right] = \frac{\cos\theta}{r^2 \sin\theta} E_0 e^{i(kr-\omega t)}$$

Which we don't have any good way of making zero.

And $\nabla \times \vec{B} = \frac{1}{c^2}\partial \vec{E}/\partial t$ doesn't go super well, either.

So now what? We could try adding more terms to E + B in such a way as to square things up. But there's a clever alternative.

We were trying to construct E + B that solve

$$\nabla^2 \vec{E} = \mu_0\varepsilon_0 \partial^2\vec{E}/\partial t^2 \quad \text{and} \quad \nabla^2 \vec{B} = \mu_0\varepsilon_0 \partial^2\vec{B}/\partial t^2$$

And you know what? In the Lorentz gauge, $\vec{A}$ + V satisfy the same equations when $\vec{J}$ and ρ are zero.

$$-\nabla^2 V + \mu_0\varepsilon_0 \frac{\partial^2 V}{\partial t^2} = \rho/\varepsilon_0 \Rightarrow \nabla^2 V = \mu_0\varepsilon_0 \partial^2 V/\partial t^2 \quad (\text{if } \rho = 0)$$

$$-\nabla^2 \vec{A} + \mu_0\varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = \mu_0 \vec{J} \Rightarrow \nabla^2 \vec{A} = \mu_0\varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} \quad (\text{if } \vec{J} = 0)$$

So let's guess the same spherical form for $\vec{A}$ (all that has to satisfy is the wave equation, which it does, though that takes some grinding)

$$\vec{A} = \frac{C}{r} e^{i(kr - \omega t)} \hat{k}$$

And then use $\nabla \times \vec{A} = \vec{B}$ to find $\vec{B}$, and $\vec{\nabla} \times \vec{B} = \mu_0\varepsilon_0 \partial\vec{E}/\partial t$ to recover $\vec{E}$.

Any E or B that we get from an A and/or V that satisfy the above differential equations, is guaranteed to satisfy the Maxwell eqns, so we cut right to the results:

$$\vec{B} = C\left(-ik/r + \frac{1}{r^2}\right) \sin\theta \, e^{i(kr-\omega t)} \hat{\phi}$$

$$\vec{E} = \frac{Cc^2}{i\omega}\left[\left(\frac{2ik}{r^2} - \frac{2}{r^3}\right)\cos\theta \, e^{i(kr-\omega t)} \hat{r} + \left(\frac{k^2}{r} + \frac{ik}{r^2} + \frac{1}{r^3}\right)\sin\theta \, e^{i(kr-\omega t)} \hat{\theta}\right]$$

Fields which look a little like [slide]

Now, we constructed this solution to satisfy the wave equation, meaning that it can exist: but not that it actually does exist. We've not seen or said anything about the source of these fields.

For a little bit of physical insight, consider some limits.

r very large is called the radiation zone, for as we get far from a source, the dominant fields should be those from any radiation that exists. For large r the above yield:

$$\vec{B} = \frac{-ikC}{r} \sin\theta \, e^{i(kr-\omega t)} \hat{\phi}, \quad \vec{E} = \frac{Cc^2k^2}{i\omega r} \sin\theta \, e^{i(kr-\omega t)} \hat{\theta}$$

Now think about source

Those $1/r$ dependences are a good thing. Given a source radiating with constant power, the radiation intensity I (power per area) should decrease as $1/r^2$. And $I = |\vec{S}| \propto \vec{E} \times \vec{B}$, which in this case would give a $1/r^2$ dependence.

Now what about for r very small, the so-called near zone? At that point, the dominant fields should have a functional form reflecting the charge distribution acting as a source. For small r, we have

$$\vec{E} \approx \frac{Cc^2}{i\omega} e^{i(kr-\omega t)} \left[-\frac{2}{r^3} \cos\theta \, \hat{r} + \frac{1}{r^3} \sin\theta \, \hat{\theta} \right]$$

That should look familiar. The bracketed part is essentially the field made by an electric dipole, and we will eventually obtain the tools necessary to explicitly calculate the fields made by an oscillating dipole. We will get exactly this spherical wave.