#### Advanced Engineering Mathematics

Homework Six Solutions

Fourier Transform: Sine/Cosine Transforms, Common Transforms, Convolution and Green's Functions

Text: 11.7-11.9

Lecture Notes : 11-12

Lecture Slides: 5

Quote of Homework Six Solutions

Aside : You know that guy from *Meet the Parents* who wanted to marry the daughter of the crazy dude. You know the human lie detector? Well, yeah his future son-in-law, that guy. What was his last name again? Well, let's call it X.

Blow the X up!

E-Days Fireworks - Colorado School of Mines Students (19?? - ????)

#### 1. Fourier Transforms of Symmetric Functions

Let,

(1.1) 
$$f_c(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}(\omega) \cos(\omega x) d\omega \qquad \qquad \hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos(\omega x) dx$$

(1.2) 
$$f_s(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}(\omega) \sin(\omega x) d\omega \qquad \qquad \hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(\omega x) dx$$

be the definitions for the Fourier cosine and Fourier sine transform pairs, respectively.

## 1.1. Symmetry. Show that $f_c(x)$ and $\hat{f}_c(\omega)$ are even functions and that $f_s(x)$ and $\hat{f}_s(\omega)$ are odd functions.<sup>1</sup>

In  $f_c(x)$ ,  $\cos(\omega x)$  is the only function of x. Thus,  $f_c(-x) = f_c(x)$ , which is the definition of an even function. Similarly,  $\hat{f}_c(-\omega) = \hat{f}_c(\omega)$ . On the other hand, in  $f_s(x)$ ,  $\sin(\omega x)$  is the only function of x. Thus,  $f_s(-x) = -f_s(x)$ , which is the definition of an odd function. Similarly,  $\hat{f}_s(-\omega) = -\hat{f}_s(\omega)$ .

1.2. Derivation from Fourier Transform. Recall the complex Fourier transform,

(1.3) 
$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega, \quad \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx.$$

Show that if we assume that f(x) is an even function then (1.3) defines the transform pair given by (1.1). Also, show that if f(x) is an odd function then (1.3) defines the transform pair given by (1.2).<sup>2</sup>

It might be best to start from the following form,

(1.4) 
$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v) e^{-i\omega v} dv \right] e^{i\omega x} d\omega$$

<sup>&</sup>lt;sup>1</sup>Thus, if an input function has an even or odd symmetry then the transformed function shares the same symmetry.

 $<sup>^{2}</sup>$ Thus, if an input function has symmetry then the Fourier transform is real-valued and reduced to a since or cosine transform.

which is what defines the Fourier transform pair (1.3). For the case where f(-x) = f(x) we have that  $\hat{f}(-\omega) = \hat{f}(\omega)$  and the following reduction,

(1.5) 
$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v) e^{-i\omega v} dv \right] e^{i\omega x} d\omega$$

(1.6) 
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v) \left( \cos(v\omega) - i\sin(v\omega) \right) dv \right] e^{i\omega x} d\omega$$

(1.7) 
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[ \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(v) \cos(v\omega) dv \right] e^{i\omega x} d\omega$$

(1.8) 
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[ \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(v) \cos(v\omega) dv \right] \left( \cos(\omega x) + i \sin(\omega x) \right) d\omega$$

(1.9) 
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[ \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(v) \cos(v\omega) dv \right] \cos(\omega x) d\omega$$

(1.10) 
$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \left[ \sqrt{\frac{2}{\pi}} \int_0^\infty f(v) \cos(v\omega) dv \right] \cos(\omega x) d\omega,$$

which defines the transform pair (1.1). For the case where f(-x) = -f(x) we have that  $\hat{f}(-\omega) = -\hat{f}(\omega)$  and the following reduction,

(1.11) 
$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v) e^{-i\omega v} dv \right] e^{i\omega x} d\omega$$

(1.12) 
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v) \left( \cos(v\omega) - i\sin(v\omega) \right) dv \right] e^{i\omega x} d\omega$$

(1.13) 
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[ -i\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(v) \sin(v\omega) dv \right] e^{i\omega x} d\omega$$

(1.14) 
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[ -i\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(v)\sin(v\omega)dv \right] \left(\cos(\omega x) + i\sin(\omega x)\right) d\omega$$

(1.15) 
$$= i \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[ -i \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(v) \sin(v\omega) dv \right] \sin(\omega x) d\omega$$

(1.16) 
$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \left[ \sqrt{\frac{2}{\pi}} \int_0^\infty f(v) \sin(v\omega) dv \right] \sin(\omega x) d\omega,$$

which defines the transform pair (1.2). From all of this we conclude that,

- If a f has symmetry then  $\hat{f}$  has the same symmetry.
- If f has an even symmetry then (1.3) reduces to (1.1).
- If f has an odd symmetry then (1.3) reduces to (1.2).

## 1.3. Even and Odd Finite Pulses. Given,

(1.17) 
$$f(x) = \begin{cases} A, & 0 < x < a \\ 0, & \text{otherwise} \end{cases}, \quad A, a \in \mathbb{R}^+$$

Plot the even and odd extensions of f.



1.4. Symmetric Transforms. Find the Fourier cosine and sine transforms of f.

$$\hat{f}_{c}(\omega) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos(\omega x) dx$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{a} A \cos(\omega x) dx = \sqrt{\frac{2}{\pi}} \left[\frac{A}{\omega} \sin(\omega x)\right]_{0}^{a}$$

$$\hat{f}_{c}(\omega) = \sqrt{\frac{2}{\pi}} \left(\frac{A \sin(a\omega)}{\omega}\right)$$

$$\hat{f}_{s}(\omega) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin(\omega x) dx$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{a} A \sin(\omega x) dx = \sqrt{\frac{2}{\pi}} \left[\frac{-A}{\omega} \cos(\omega x)\right]_{0}^{a}$$

$$\hat{f}_{s}(w) = -\sqrt{\frac{2}{\pi}} \left(\frac{A(\cos(aw) - 1)}{w}\right)$$

Both transforms have a hole at  $\omega = 0$ . However, the limit for each is well defined at  $\omega = 0$  and consequently this removable discontinuity can be filled in by the associated limit. That is,

(1.18) 
$$\hat{f}_c(\omega) = \begin{cases} \sqrt{\frac{2}{\pi} \frac{A \sin(a\omega)}{\omega}}, & \omega \neq 0\\ \sqrt{\frac{2}{\pi} Aa}, & \omega = 0 \end{cases}$$

A similar definition can be constructed for  $\hat{f}_s(\omega)$ .

1.5. Integral Trick. Using the Fourier cosine transform show that  $\int_{-\infty}^{\infty} \frac{\sin(\pi\omega)}{\pi\omega} d\omega = 1.$ 

We consider the previous results, which says that  $\mathfrak{F}^{-1}\left\{\hat{f}_{c}(\omega)\right\} = f(x)$ . That is,

(1.19) 
$$\mathfrak{F}^{-1}\left\{\hat{f}_{c}(\omega)\right\} = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \sqrt{\frac{2}{\pi}} \frac{A\sin(a\omega)}{\omega} \cos(\omega x) d\omega$$
(1.20) 
$$= \frac{2}{\pi} \int_{0}^{\infty} \frac{A\sin(a\omega)}{\omega} \cos(\omega x) d\omega$$

(1.20) 
$$= -\frac{\pi}{\omega} \int_0^{\omega} \frac{\partial \omega(\omega x)}{\omega} \cos(\omega x) dx$$

$$(1.21) = f(x).$$

Since this should work for any x value we have that  $f(0) = A^{3}$  and,

$$(1.22) f(0) = A$$

(1.23) 
$$= \frac{2A}{2} \int_{-\infty}^{\infty} \frac{\sin(a\omega)}{\cos(\omega \cdot 0)} \cos(\omega \cdot 0) d\omega$$

(1.24) 
$$= \frac{A}{\pi} \int_{-\infty}^{\infty} \frac{\sin(a\omega)}{\omega} d\omega,$$

which implies that,

(1.25) 
$$\int_{-\infty}^{\infty} \frac{\sin(a\omega)}{\omega} d\omega = \pi.$$

#### 2. Sine and Cosine Transforms

Calculate the following Fourier sine/cosine transformations. Include any domain restrictions.

<sup>&</sup>lt;sup>3</sup>Really f(x) isn't defined at this point but we know that Fourier methods would just average the right and hand left hand limits at this point, which is A.

## 2.1. Forward Cosine Transform. $\mathfrak{F}_c(e^{-ax}), a \in \mathbb{R}^+$

The following integral can be done either by integration-by-parts or imaginary-exponential functions. Here we quote the anti-derivative found in homework 4 problem 1 part 3 equation 5.

(2.1) 
$$\hat{f}_c(w) = \mathfrak{F}_c(e^{-ax})$$

(2.2) 
$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos(\omega x) dx =$$

(2.3) 
$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos(wx) dx$$

(2.4) 
$$= \sqrt{\frac{2}{\pi}} \left[ \frac{e^{-ax}}{a^2 + \omega^2} \left( \omega \sin(\omega x) - a \cos(\omega x) \right) \right]_0^\infty$$

(2.5) 
$$= \sqrt{\frac{2}{\pi}} \left\{ 0 - \frac{1}{a^2 + \omega^2} \left( \omega \cdot \sin(0) - a \cos(0) \right) \right\}$$

(2.6) 
$$= \sqrt{\frac{2}{\pi} \frac{a}{a^2 + \omega^2}}, \omega \in \mathbb{R}.$$

# 2.2. Inverse Cosine Transform. $\mathfrak{F}_c^{-1}\left(\frac{1}{1+\omega^2}\right)$

The following integral cannot be formally calculated without using the so-called *residue calculus*. However, using the duality of Fourier transform pairs we can relate the integral to the preceding problem. Notice that,

(2.7) 
$$\mathfrak{F}_{c}^{-1}\left\{\frac{1}{1+\omega^{2}}\right\} = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \frac{1}{1+\omega^{2}} \cos(\omega x) d\omega$$

(2.8) 
$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{1+\omega^2} \cos(\omega x) d\omega,$$

looks like the inverse transform of  $\sqrt{\frac{\pi}{2}}\hat{f}_c(\omega) = \frac{a}{a^2 + \omega^2}$  where a = 1. Thus, the inverse Fourier transform of  $\frac{1}{1 + \omega^2}$  is given by  $\sqrt{\frac{\pi}{2}}e^{-x}$  for x > 0.

# 2.3. Forward Sine Transform. $\mathfrak{F}_s(e^{-ax}), \ a \in \mathbb{R}^+$

The following integral can be done either by integration-by-parts or imaginary-exponential functions. Here we quote the anti-derivative found in homework 4 problem 1 part 3 equation 6.

(2.9) 
$$\mathfrak{F}_s\left\{e^{-ax}\right\} = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(\omega) \sin(\omega x) dx$$

(2.10) 
$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin(\omega x) dx$$

(2.11) 
$$= \sqrt{\frac{2}{\pi}} \left[ \frac{e^{-ax}}{a^2 + \omega^2} \left( -a\sin(\omega x) - \omega\cos(\omega x) \right) \right]_0^\infty$$

(2.12) 
$$= \sqrt{\frac{2}{\pi}} \left[ 0 - \frac{1}{a^2 + \omega^2} \left\{ -a\sin(0) - \omega\cos(0) \right\} \right]$$

(2.13) 
$$= \sqrt{\frac{2}{\pi}} \frac{\omega}{a^2 + \omega^2}, \omega \in \mathbb{R}.$$

2.4. Inverse Sine Transform.  $\mathfrak{F}_s^{-1}\left(\sqrt{\frac{2}{\pi}}\frac{\omega}{a^2+\omega^2}\right), \ a \in \mathbb{R}^+$ 

The following integral cannot be formally calculated without using the so-called *residue calculus*. However, using the duality of Fourier transform pairs we can relate the integral to the preceding problem.

(2.14) 
$$\mathfrak{F}_s^{-1}\left\{\frac{\omega}{a^2+\omega^2}\right\} = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(w)\sin(\omega x)d\omega$$

(2.15) 
$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\omega \sin(\omega x)}{a^2 + \omega^2} d\omega$$

$$(2.16) \qquad \qquad = \sqrt{\frac{2}{\pi}} \left[ \sqrt{\frac{\pi}{2}} e^{-x} \right]$$

(2.17) 
$$= e^{-x}, x > 0$$

### 3. Fourier Transforms

Calculate the following transforms. Include any domain restrictions.

3.1. **Dirac-Delta.**  $\mathfrak{F}\{f\}$  where  $f(x) = \delta(x - x_0), \ x_0 \in \mathbb{R}^4$ 

(3.1) 
$$\mathfrak{F}\left\{f\right\} = \delta(\mathfrak{x} - \mathfrak{x}_{\circ}) = \hat{f}(\omega)$$

(3.2) 
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x - x_0) e^{-i\omega x} dx$$

(3.3)  

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x - x_0) e^{-i\omega x_0} dx$$
(3.4)  

$$= \frac{e^{-i\omega x_0}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x - x_0) dx$$

(3.4)  

$$= \frac{e^{-i\omega x_0}}{\sqrt{2\pi}} \int_{-\infty} \delta(t) dt$$
(3.5)  

$$= \frac{e^{-i\omega x_0}}{\sqrt{2\pi}}$$

3.2. Decaying Exponential Function.  $\mathfrak{F}\{f\}$  where  $f(x) = e^{-k_0|x|}, k_0 \in \mathbb{R}^+$ .

$$\begin{split} f(x) &= \begin{cases} e^{-k_0 x} & x > 0\\ e^{k_0 x} & x < 0 \end{cases} \\ \hat{f}(w) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 e^{k_0 x} e^{iwx} dx + \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-k_0 x} e^{iwx} dx = \\ &= \frac{1}{\sqrt{2\pi}} \left[ \frac{e^{(k_0 + iw)x}}{k_0 + iw} \Big|_{-\infty}^0 + \frac{e^{(iw - k_0)}}{iw - k_0} \Big|_0^\infty \right] = \\ &= \frac{1}{\sqrt{2\pi}} \left[ \frac{1}{k_0 + iw} - \frac{1}{iw - k_0} \right] = \frac{1}{\sqrt{2\pi}} \left[ \frac{-2k_0}{-w^2 - k_0^2} \right] = \\ &= \sqrt{\frac{2}{\pi}} \left( \frac{k_0}{w^2 + k_0^2} \right) \end{split}$$

3.3. Even Combination of Dirac-Deltas.  $\mathfrak{F}^{-1}\left\{\hat{f}\right\}$  where  $\hat{f}(\omega) = \frac{1}{2}\left(\delta(\omega + \omega_0) + \delta(\omega - \omega_0)\right), \ \omega_0 \in \mathbb{R}.$ 

$$f(x) = \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)\right] e^{i\omega x} d\omega$$
$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{i\omega_0 x} + e^{-i\omega_0 x}}{2}\right] = \frac{1}{\sqrt{2\pi}} \cos(\omega_0 x)$$

<sup>&</sup>lt;sup>4</sup>Here the  $\delta$  is the so-called Dirac, or continuous, delta function. This isn't a function in the true sense of the term but instead what is called a generalized function. The details are unimportant and in this case we care only that this Dirac-delta function has the property  $\int_{-\infty}^{\infty} \delta(x-x_0)f(x)dx = f(x_0)$ . For more information on this matter consider http://en.wikipedia.org/wiki/Dirac\_delta\_function. To drive home that this function is strange, let me spoil the punch-line. The sampling function  $f(x) = \operatorname{sinc}(ax)$  can be used as a definition for the Delta function as  $a \to 0$ . So can a bell-curve probability distribution. Yikes!

3.4. Odd Combination of Dirac-Deltas.  $\mathfrak{F}^{-1}\left\{\hat{f}\right\}$  where  $\hat{f}(\omega) = \frac{1}{2}\left(\delta(\omega + \omega_0) - \delta(\omega - \omega_0)\right), \ \omega_0 \in \mathbb{R}$ .

$$f(x) = \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)\right] e^{iwx} dw =$$
$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{i\omega_0 x} - e^{-i\omega_0 x}}{2}\right] = \frac{i}{\sqrt{2\pi}} \sin(\omega_0 x)$$

3.5. Horizontal Shifts. Find  $\hat{f}(\omega)$  where  $f(x+c), c \in \mathbb{R}$ .

$$\mathfrak{F}\{f(x+c)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x+c)e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)e^{-i\omega(u-c)} du = e^{i\omega c} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)e^{-i\omega u} du = e^{i\omega c} \hat{f}(\omega)$$
4. CONVOLUTION INTEGRALS

The convolution h of two functions f and g is defined as<sup>5</sup>,

(4.1) 
$$h(x) = (f * g)(x) = \int_{-\infty}^{\infty} f(p)g(x-p)dp = \int_{-\infty}^{\infty} f(x-p)g(p)dp.$$

4.1. Fourier Transforms of Convolutions. Show that  $\mathfrak{F} \{f * g\} = \sqrt{2\pi} \mathfrak{F} \{f\} \mathfrak{F} \{g\}$ .

$$\begin{split} \mathfrak{F}\{f*g\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(p)g(x-p)dp \right] e^{-i\omega x} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p)g(x-p)e^{-i\omega x} dx dp = \qquad \begin{array}{c} \operatorname{let} x-p=q \\ \Rightarrow x=q+p \end{array} \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(p)g(q)e^{-i\omega(p+q)} dq dp = \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(p)e^{-i\omega p} dp \cdot \int_{-\infty}^{\infty} g(q)e^{-i\omega q} dq = \\ &= \left[ \mathfrak{F}\{f\}\sqrt{2\pi}\mathfrak{F}\{g\} \right] = \sqrt{2\pi}\mathfrak{F}\{f\}\mathfrak{F}\{g\} \end{split}$$

4.2. Simple Convolution. Find the convolution h(x) = (f \* g)(x) where  $f(x) = \delta(x - x_0)$  and  $g(x) = e^{-x}$ .

$$(f * g)(x) = \int_{-\infty}^{\infty} \delta(p - x_0) e^{-(x-p)} dp = e^{-(x-x_0)}$$

5. SIMPLE GREEN'S FUNCTIONS

Given the ODE,

(5.1)  $y' + y = f(x), \quad -\infty < x < \infty.$ 

5.1. Delta Forcing. Calculate the frequency response, or what is sometimes called the steady-state transfer function, associated with (5.1).<sup>7</sup>

The frequency response function is the Fourier transform of the solution to (5.1) where  $f(x) = \delta(x)$ . It represents how (5.1) wants to respond to the most primitive of forces, the delta force, in the frequency domain. To differentiate it from y we write down (5.1) with y's replaced by g's and  $f(x) = \delta(x)$  and solve the problem in the Fourier domain.

$$\begin{split} \mathfrak{F}\{g'+g\} &= i\omega \hat{g} + \hat{g} \\ &= \hat{g}(1+i\omega) \\ &= \mathfrak{F}\{\delta(x)\} = \frac{1}{\sqrt{2\pi}} \implies \hat{g}(\omega) = \frac{1}{\sqrt{2\pi} \left(i\omega + 1\right)}, \end{split}$$

 $<sup>^5\</sup>mathrm{Here}$  wee keep the same notation as Kreysig pg. 523

<sup>&</sup>lt;sup>6</sup>The point here is that while the Fourier transform of a linear combination is the linear combination of Fourier transforms, the Fourier transform of a product is a convolution integral. Well, at least that's something.

<sup>&</sup>lt;sup>7</sup>This function is a representation of how the system responds to the most primitive force,  $\delta(x)$ , in the Fourier domain.

where  $\hat{g}(\omega)$  is called the frequency response function associated with the ODE (5.1).

5.2. Inversion. Calculate the Green's function associated with (5.1).<sup>8</sup>

The Green's function for (5.1) is given by the inverse transform of the frequency response function and represents how (5.1) behaves with a simple force like a delta function. The delta function is considered simple in the sense that it embodies an initial impulse of force given to the system. Once this simple impulse of force is understood then more complicated forces can be used. To find the Green's function we take an inverse Fourier transform. Let  $= \hat{g}(\omega) = \left(\frac{1}{\sqrt{2\pi}(i\omega+1)}\right)$  then

(5.2) 
$$\mathfrak{F}^{-1}\{\hat{g}\} = g(x)$$

(5.3) 
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi} \left(i\omega + 1\right)} e^{i\omega x} d\omega$$

(5.4) 
$$= \begin{cases} e^{-x}, & x > 0\\ 0, & \text{otherwise} \end{cases}$$

by the use of Table III element 5 on page 531. Hence, g(x) is the Green's function for the ODE (5.1).

5.3. Solutions as Convolution Integrals. Using convolution find the steady-state solution to the (5.1).<sup>9</sup> Once a Green's function is known we have the following logic,

(5.5) 
$$\mathfrak{F}\left\{y'+y\right\} = \hat{y}(\omega)\left(i\omega+1\right) = \mathfrak{F}\left\{f\right\} = \hat{f}(\omega)$$

implies that,

(5.6) 
$$\hat{y}(\omega) = \frac{\hat{f}(\omega)}{i\omega+1} = \sqrt{2\pi}\hat{f}(\omega)\hat{g}(\omega)$$

Thus by the convolution integral in the previous problem we have,

(5.7) 
$$\mathfrak{F}^{-1}\left\{y\right\} = \mathfrak{F}^{-1}\left\{\sqrt{2\pi}\hat{f}(\omega)\hat{g}(\omega)\right\}$$

$$(5.8) \qquad \qquad = (f * g)(x)$$

(5.9) 
$$= \int_{-\infty}^{\infty} f(p)g(x-p)dp$$

(5.10) 
$$= \int_0^\infty f(p)e^{-(x-p)}dp.$$

Therefore, we have that the steady-state solution is  $y(x) = e^{-x} \int_0^\infty f(p)e^p dp$  for for an arbitrary f(x). As a quick sanity check we can take  $f(x) = \delta(x)$ , which immediately give that  $y(x) = e^{-x} \int_0^\infty \delta(p)e^p dp = e^{-x}$  as expected from this problem as well as the previous problem on convolution.

<sup>&</sup>lt;sup>8</sup>The Green's function is just the inverse of the frequency response function and is a representation of how the system would like to respond to the primitive,  $\delta(x)$ , force in the original domain.

<sup>&</sup>lt;sup>9</sup>The key point of these three steps is that, if you can determine how a linear differential equation responds to simple forcing then the general solution can be represented in terms of a convolution integral containing the Green's function. This gives us a general method for solving inhomogeneous differential equations. However, finding the Green's function for a DE is generally nontrivial.