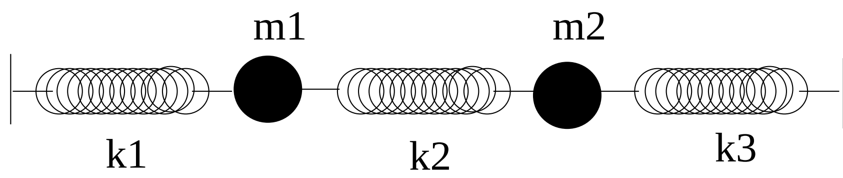


Figure 1.10: Two masses coupled by a spring and attached to walls.

This splitting of the degenerate frequency by an external magnetic field is called the Zeeman effect, after its discoverer Pieter Zeeman was born in May 1865, at Zonnemaire, a small village in the isle of Schouwen, Zeeland, The Netherlands. Zeeman was a student of the great physicists Onnes and Lorentz in Leyden. He was awarded the Nobel Prize in Physics in 1902. Zeeman succeeded Van der Waals (another Nobel prize winner) as professor and director of the Physics Laboratory in Amsterdam in 1908. In 1923 a new laboratory was built for Zeeman that included a quarter-million kilogram block of concrete for vibration free measurements.

We could continue the analysis by plugging these frequencies back into the amplitude equations 1.1.35. As an exercise, do this and show that the motion of the electron (and hence the electric field) is circularly polarized in the direction perpendicular to the magnetic field.

1.2 Two Coupled Masses

With only one mass and one spring, the range of motion is somewhat limited. There is only one characteristic frequency $\omega_0^2 = \frac{k}{m}$ so in the absence of damping, the transient (unforced) motions are all of the form $\cos(\omega_0 t + \Delta)$.

Now let us consider a slightly more general kind of oscillatory motion. Figure 1.10 shows two masses (m_1 and m_2) connected to fixed walls with springs k_1 and k_3 and connected to one another by a spring k_2. To derive the equations of motion, let's focus attention on one mass at a time. We know that for any given mass, say m_i (whose displacement from equilibrium we label x_i) it must be that

$$m_i \ddot{x}_i = F_i \tag{1.2.1}$$

where F_i is the total force acting on the ith mass. No matter how many springs and masses we have in the system, the force applied to a given mass must be transmitted by the two springs it is connected to. And the force each of these springs transmits is governed by the extent to which the spring is compressed or extended.

Referring to Figure 1.10, spring 1 can only be compressed or extended if mass 1 is displaced from its equilibrium. Therefore the force applied to m_1 from k_1 must be $-k_1 x_1$, just as before. Now, spring 2 is compressed or stretched depending on whether $x_1 - x_2$ is positive or not. For instance, suppose both masses are displaced to the right (positive x_i) with mass 1 being displaced more than mass 2. Then spring 2 is compressed relative to its equilibrium length and the force on mass 1 will in the negative x direction so as to restore the mass to its equilibrium position. Similarly, suppose both masses are displaced to the right, but now with mass 2 displaced more than mass 1, corresponding to spring 2 being stretched. This should result in a force on mass 1 in the positive x direction since the mass is being pulled away from its equilibrium position. So the proper expression of Hooke's law in any case is

$$m_1 \ddot{x}_1 = -k_1 x_1 - k_2(x_1 - x_2). \tag{1.2.2}$$

And similarly for mass 2

$$m_2 \ddot{x}_2 = -k_3 x_2 - k_2(x_2 - x_1). \tag{1.2.3}$$

These are the general equations of motion for a two mass/three spring system. Let us simplify the calculations by assuming that both masses and all three springs are the same. Then we have

$$\begin{aligned} \ddot{x}_1 &= -\frac{k}{m} x_1 - \frac{k}{m}(x_1 - x_2) \\ &= -\omega_0^2 x_1 - \omega_0^2 (x_1 - x_2) \\ &= -2\omega_0^2 x_1 + \omega_0^2 x_2. \end{aligned} \tag{1.2.4}$$

and

$$\begin{aligned} \ddot{x}_2 &= -\frac{k}{m} x_2 - \frac{k}{m}(x_2 - x_1) \\ &= -\omega_0^2 x_2 - \omega_0^2 (x_2 - x_1) \\ &= -2\omega_0^2 x_2 + \omega_0^2 x_1. \end{aligned} \tag{1.2.5}$$

Assuming trial solutions of the form

$$x_1 = Ae^{i\omega t} \tag{1.2.6}$$

$$x_2 = Be^{i\omega t} \tag{1.2.7}$$

we see that

$$(-\omega^2 + 2\omega_0^2)A = \omega_0^2 B \tag{1.2.8}$$

$$(-\omega^2 + 2\omega_0^2)B = \omega_0^2 A. \tag{1.2.9}$$

Substituting one into the other we get

$$A = \frac{\omega_0^2}{2\omega_0^2 - \omega^2} B, \tag{1.2.10}$$

and therefore

$$(2\omega_0^2 - \omega^2)B = \frac{\omega_0^4}{2\omega_0^2 - \omega^2}B. \tag{1.2.11}$$

This gives an equation for ω^2

$$(2\omega_0^2 - \omega^2)^2 = \omega_0^4. \tag{1.2.12}$$

There are two solutions of this equation, corresponding to $\pm\omega_0^2$ when we take the square root. If we choose the plus sign, then

$$2\omega_0^2 - \omega^2 = \omega_0^2 \Rightarrow \omega^2 = \omega_0^2. \tag{1.2.13}$$

On the other hand, if we choose the minus sign, then

$$2\omega_0^2 - \omega^2 = -\omega_0^2 \Rightarrow \omega^2 = 3\omega_0^2. \tag{1.2.14}$$

We have discovered an important fact: spring systems with two masses have two characteristic frequencies. We will refer to the frequency $\omega^2 = 3\omega_0^2$ as "fast" and $\omega^2 = \omega_0^2$ as "slow". Of course these are relative terms. Now that we have the frequencies we can investigate the amplitude. First, since

$$A = \frac{\omega_0^2}{2\omega_0^2 - \omega^2}B, \tag{1.2.15}$$

we have for the slow mode ($\omega = \omega_0$)

$$A = B, \tag{1.2.16}$$

which corresponds to the two masses moving in phase with the same amplitude. On the other hand, for the fast mode

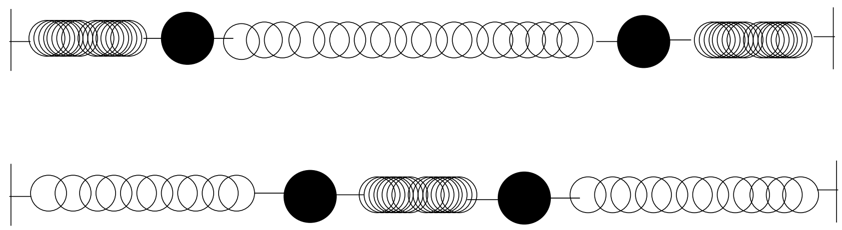
$$A = -B. \tag{1.2.17}$$

For this mode, the amplitudes of the two mass' oscillation are the same, but they are out of phase. These two motions are easy to picture. The slow mode corresponds to both masses moving together, back and forth, as in Figure 1.11 (bottom). The fast mode corresponds to the two masses oscillating out of phase as in Figure 1.11 (top).

1.2.1 A Matrix Appears

There is a nice way to simplify the notation of the previous section and to introduce a powerful mathematical at the same time. Let's put the two displacements together into a vector. Define a vector $\mathbf{u}$ with two components, the displacements of the first and second mass:

$$\mathbf{u} = \begin{bmatrix} Ae^{i\omega t} \\ Be^{i\omega t} \end{bmatrix} = e^{i\omega t}\begin{bmatrix} A \\ B \end{bmatrix}. \tag{1.2.18}$$



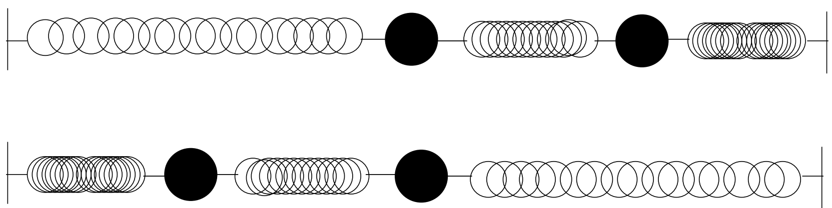


Figure 1.11: With two coupled masses there are two characteristic frequencies, one "slow" (bottom) and one "fast" (top).

We've already seen that we can multiply any solution by a constant and still get a solution, so we might as well take A and B to be equal to 1. So for the slow mode we have

$$\mathbf{u} = e^{i\omega_0 t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{1.2.19}$$

while for the fast mode we have

$$\mathbf{u} = e^{i\sqrt{3}\omega_0 t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \tag{1.2.20}$$

Notice that the amplitude part of the two modes

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{1.2.21}$$

are **orthogonal**. That means that the dot product of the two vectors is zero: $1 \times 1 + 1 \times (-1) = 0$.[6] As we will see in our discussion of linear algebra, this means that the two vectors point at right angles to one another. This orthogonality is an absolutely fundamental property of the natural modes of vibration of linear mechanical systems.

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$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} \equiv [1,1] \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 \cdot 1 - 1 \cdot 1 = 0.$$

1.2.2 Matrices for two degrees of freedom

The equations of motion are (see Figure 1.10):

$$m_1\ddot{x}_1 + k_1 x_1 + k_2(x_1 - x_2) = 0 \tag{1.2.22}$$

$$m_2\ddot{x}_2 + k_3 x_2 + k_2(x_2 - x_1) = 0. \tag{1.2.23}$$

We can write these in matrix form as follows.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{1.2.24}$$

Or, defining a mass matrix

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \tag{1.2.25}$$

and a "stiffness" matrix

$$K = \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix} \tag{1.2.26}$$

we can write the matrix equation as

$$M\ddot{\mathbf{u}} + K\mathbf{u} = \mathbf{0} \tag{1.2.27}$$

where

$$\mathbf{u} \equiv \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{1.2.28}$$

This is much cleaner than writing out all the components and has the additional advantage that we can add more masses/springs without changing the equations, we just have to incorporate the additional terms into the definition of M and K.

Notice that the mass matrix is always invertible since it's diagonal and all the masses are presumably nonzero. Therefore

$$M^{-1} = \begin{bmatrix} {m_1}^{-1} & 0 \\ 0 & {m_2}^{-1} \end{bmatrix}. \tag{1.2.29}$$

So we can also write the equations of motion as

$$\ddot{\mathbf{u}} + M^{-1}K\mathbf{u} = \mathbf{0}. \tag{1.2.30}$$

And it is easy to see that

$$M^{-1}K = \begin{bmatrix} \frac{k_1+k_2}{m_1} & \frac{-k_2}{m_1} \\ \frac{-k_2}{m_2} & \frac{k_2+k_3}{m_2} \end{bmatrix}.$$