Poynting theorem

- calc rate of work done by fields.

consider a single charge q

work done $= \int \vec{F}\cdot d\ell$

$$= \int q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)\cdot\vec{v}\,dt$$

$$= \int q\,\vec{E}\cdot\vec{v}\,dt$$ work is done only by E field.

express in terms of volume

$$q = \rho\, d^3x$$
$$\rho\vec{v} = \vec{J}$$

$\therefore$ rate of work done (power)

$$\frac{dW}{dt} = \int \rho\,\vec{E}\cdot\vec{v}\,d^3x = \int_V \vec{E}\cdot\vec{J}\,d^3x$$

next, express in terms of fields only.

calc. $E\cdot\vec{J}$

start with

$$\nabla\times\vec{H} - \frac{1}{c}\frac{\partial\vec{D}}{\partial t} = \frac{4\pi}{c}\vec{J}_f$$

$$\therefore\ \vec{E}\cdot\vec{J} = \frac{c}{4\pi}\vec{E}\cdot(\nabla\times\vec{H}) - \frac{1}{4\pi}\vec{E}\cdot\frac{\partial\vec{D}}{\partial t}$$

vector ID: $\nabla\cdot(\vec{E}\times\vec{H}) = \vec{H}\cdot(\nabla\times\vec{E}) - \vec{E}\cdot(\nabla\times\vec{H})$

Poynting vector example:

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \qquad \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ E_x & 0 & 0 \end{vmatrix}$$

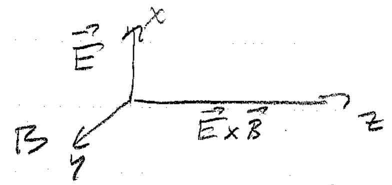
$$-\partial_z E\,\hat{y} = +\hat{y}\,E_0 \sin(\,)\,k$$

plane wave in vacuum:

$$B = -\int E_0 k c \sin(\,)\,dt$$

$$\vec{E}(z,t) = \hat{x} E_0 \cos(kz - \omega t) \qquad = E_0 \cos(\,)$$

$$\vec{B}(z,t) = \hat{y} B_0 \cos(kz - \omega t) \qquad \underline{B_0 = E_0}$$



in vacuum: $\vec{S} = \frac{c}{4\pi}(\vec{E} \times \vec{H})$ — in vacuum $H_0 = B_0 = E_0$

$$= \frac{c}{4\pi} E_0^2 \cos^2(kz - \omega t)\,\hat{z}$$

power flows in direction of wave $(\vec{k})$

time average:

$$\bar{S} = c \cdot \frac{1}{8\pi} E_0^2 = c \cdot \text{energy density.}$$

$$= \frac{\text{energy}}{\text{time}} \cdot \frac{1}{\text{area}} = \text{intensity}$$

SI $\vec{S} \equiv \vec{E} \times \vec{H} = \frac{1}{\mu_0}\vec{E} \times \vec{B}$

$B_0 = E_0/c$ in wave $\qquad c^2 = 1/\mu_0\varepsilon_0$

$$\rightarrow \underset{\substack{\uparrow \\ \text{avg}}}{\bar{S}} = \frac{1}{2}\frac{1}{\mu_0 c} E_0^2 = \frac{1}{2}\frac{\varepsilon_0 \mu_0 c}{\mu_0} E_0^2 = \frac{1}{2} c \varepsilon_0 E^2$$

energy density in SI $\quad \frac{1}{2}\varepsilon_0 E^2, \ \frac{1}{2}\frac{1}{\mu_0}B^2$

S or I in W/m^2

u in J/m^3

convert vacuum: $E^2_{Gauss} \rightarrow 4\pi\varepsilon_0 E^2_{SI} \qquad B^2_G \rightarrow \frac{4\pi}{\mu_0} B^2_{SI}$

Discussion of the Poynting theorem: conservation of energy

Book (HM 4.6) derives

$$\frac{\partial}{\partial t}\mathcal{E} + \nabla\cdot\vec{S} + \vec{E}\cdot\vec{J}_f = 0$$

where $\mathcal{E}$ = field energy density (Helmholtz free energy)

$$= \frac{1}{8\pi}\left(\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B}\right) \rightarrow \underline{\frac{1}{8\pi}\left(\varepsilon E^2 + \frac{1}{\mu}B^2\right)}$$

isotropic medium

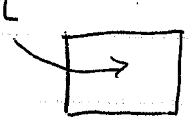
$$\vec{S} \equiv \frac{c}{4\pi}\vec{E}\times\vec{H} \quad \text{Poynting flux}$$

this represents the <u>flow</u> of energy
in simple waves $\vec{S} \rightarrow \hat{k}\cdot$ intensity

$\vec{E}\cdot\vec{J}_f$ = work done on charges by field / unit time.
(power per unit volume)

With no charges present, we have a continuity for energy density; similar to that for charge:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{J} = 0$$



note that $\vec{J} = \rho\vec{u}$

change in local ρ = difference btw inflow and outflow.
assuming no sources or sinks.
suppose we had ionization at rate W_I
- this goes in on RH side as a source.

for energy, $\vec{S}$ is the equivalent of $\vec{J}$

dimensions E^2, B^2, EH ... are energy density

$$S \sim cEH \sim \text{energy flux.}$$

in SI $\vec{S} = \vec{E} \times \vec{H}$

Work done on charges:

Lorentz force $\vec{F} = q\vec{E} + q\frac{\vec{v}}{c} \times \vec{B}$

Work done:

$$W = \int \vec{F} \cdot d\vec{l} = \int q\vec{E} \cdot d\vec{l}$$

$\vec{B}$ field does no work since $(\vec{v} \times \vec{B}) \cdot d\vec{l} = 0$

convert to integral on time: $W = \int q\vec{E} \cdot \vec{v}\, dt$

rate of work done (power)

$$\frac{dW}{dt} = q\vec{E} \cdot \vec{v} = \int \vec{E} \cdot \vec{J}\, d^3x$$

this assumes that all current density results from $\vec{E}$ (initially $\vec{J} = 0$)

$\therefore$ $\vec{E} \cdot \vec{J}$ = work done on charges / volume.

$= \frac{d}{dt} \mathcal{E}_{mech}$ rate of change of mechanical $\mathcal{E}$

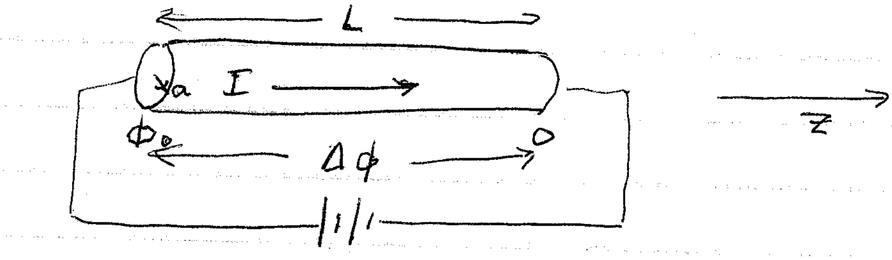
$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \vec{S} = -\vec{E} \cdot \vec{J}$$

is statement of conservation of field energy

role is a "sink" term.

Some examples from circuits, but will use $\vec{S}$ a lot shortly with EM waves.

Example: current in a resistive wire



expect $V = IR$ $\quad (V = \Delta\phi)$

$P = IV$

to use Poynting thm, find fields

$\vec{E} = -\nabla\Phi = \Delta\Phi/L$

this is a static problem: $\partial E/\partial t = 0$

find $\vec{B}$ note $\mu = 1$ so $\vec{B} = \vec{H}$

$$\nabla\times\vec{B} = \frac{1}{c}\underbrace{\frac{\partial\vec{E}}{\partial t}}_{0} + \frac{4\pi}{c}\vec{J} \qquad \vec{J} = \frac{I}{\pi a^2}\hat{z}$$

$$\vec{B} = B(r)\hat{\phi}$$

$$\int(\nabla\times B)\cdot d\vec{S} = \oint_r \vec{B}\cdot d\vec{l} = \frac{4\pi}{c}\int\vec{J}\cdot d\vec{S}$$

$$2\pi r B(r) = \frac{4\pi}{c}\frac{I}{\pi a^2}\cdot\pi r^2 \qquad r\le a$$

$$= \frac{4\pi}{c}\frac{I}{\pi a^2}\pi a^2 \qquad r\ge a$$

$$B(r) = \frac{2I}{c}\frac{r}{a^2} \qquad r\le a$$

$$= \frac{2}{c}\frac{I}{r} \qquad r\ge a$$

we want $B(a) = \dfrac{2I}{ca}$

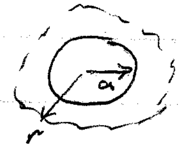
$$S = \frac{c}{4\pi}\vec{E}\times\vec{H} = \frac{c}{4\pi}E\cdot B(a)(-\hat{r})$$

$$= \frac{-c}{4\pi}E\,\frac{2I}{ca} = -\frac{VI}{2\pi aL}$$

integrate over whole surface of wire $\int\vec{S}\cdot d\vec{a} = S\cdot 2\pi aL$

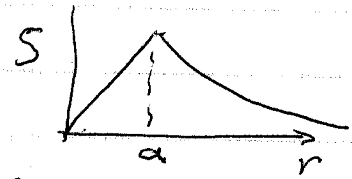
$= VI$

Note we can choose any surface:



$$S(r) = -\frac{VI}{2\pi L}\frac{r}{a^2} \qquad r \leq a$$

$$= -\frac{VI}{2\pi L}\frac{1}{r} \qquad r \geq a$$



S follows $B(r)$, peak at surface

- power is drawn from fields, dumped evenly in wire