

SECOND ORDER LINEAR EQUATIONS

Quote of Second Order Linear Equations Homework	
When I get to the bottom I go back to the top of the slide. Where I stop and turn and I go for a ride 'till I get to the bottom and I see you again.	
	The Beatles : The White Album (1968)

0. INTRODUCTION

The following homework concentrates on the second order linear equation,

$$(1) \quad a(t) \frac{d^2 y}{dt^2} + b(t) \frac{dy}{dt} + c(t)y = f(t),$$

where a, b, c are smooth functions of time and f is a piecewise smooth or distributional in nature.¹ From class we understand that the general solution to such an equation is given by

$$(2) \quad y(t) = y_h(t) + y_p(t),$$

where $y_h(t)$ is the general solution to the corresponding homogeneous problem of (1) and y_p is exactly one particular solution to (1). Moreover, we know that (1), where $f(t) \equiv 0$ defines a two-dimensional solution space, which implies that we search for homogeneous solutions of the form,

$$(3) \quad y_h(t) = c_1 y_1(t) + c_2 y_2(t), \quad c_1, c_2 \in \mathbb{R},$$

where y_1, y_2 are both solutions to the homogeneous ODE. Also, we understand that if you know one homogeneous solution to (1) then a second linearly independent solution can be found via

$$(4) \quad y_2(t) = k(t)y_1(t), \quad k(t) = \int \frac{p(t)}{[y_1(t)]^2} dt, \quad p(t) = e^{-\int (b(t)/a(t)) dt},$$

which gives the general homogeneous solution. Once this is known we can then find the particular solution via

$$(5) \quad y_p(t) = y_2 \int \frac{f(t)y_1(t)}{a(t)W(t)} dt - y_1 \int \frac{f(t)y_2(t)}{a(t)W(t)} dt, \quad W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t),$$

That being said, I need to emphasize that this is *the hard way* and that these formulae should only be used as a last resort. I say this because it is always possible to verify proposed solutions to differential equations by direct substitution. This means that if there exists a regimented system of guessing then we can exploit this to find solutions without resorting to (4) and (5). Just as before, we motivate (1) with some mathematical models stemming from Newton's second law.²

¹ We won't need the idea of a *distribution* quite yet and when we get there we won't need any theory. So, just let that word float around for a while.

² A body of mass m subject to a net force F undergoes an acceleration a that has the same direction as the force and a magnitude that is directly proportional to the force and inversely proportional to the mass, i.e., $F = ma$. Alternatively, the total force applied on a body is equal to the time derivative of linear momentum of the body.

Quote of Linear v. Nonlinear	
<p>Professor Hubert Farnsworth: Good news, everyone. You'll be making a delivery to the planet Trisol. A mysterious planet located in the mysterious depths of the Forbidden Zone.</p>	
<p>Leela: Professor, are we even allowed in the Forbidden Zone?</p>	
<p>Professor Hubert Farnsworth: Why of course. It's just a name, like the Death Zone, or the Zone of No Return. All the zones have names like that in the Galaxy of Terror.</p>	
	Futurama: My Three Suns (1999)

1. NONLINEAR OSCILLATIONS AND CONNECTIONS TO LINEAR OSCILLATIONS

In class we presented a derivation of the linear mass-spring equation. The critical assumptions were

- (1) The force due to friction is proportional to the velocity.³
- (2) The force due to the spring is proportional to the displacement of the mass from the mass-spring equilibrium state.⁴

For now we won't abandon these assumptions but we will try to achieve nonlinearity another way.

1.1. Oscillating Pendulum. Consider a mass, m , attached to one end of a rigid, but weightless, rod of length L . The other end of the rod is supported at the frictionless origin O , and the rod is free to rotate in the plane. The position of the pendulum is described by the angle, θ , between the rod and the downward vertical direction, with the counterclockwise direction taken as positive.

1.1.1. Pendulum Equation. Using the previous description, draw a diagram of the system and by appealing to Newton's second law show that the equation of motion associated with the motion on θ is given by

$$(6) \quad \frac{d^2\theta}{dt^2} + \gamma \frac{d\theta}{dt} + \omega^2 \sin(\theta) = f(t),$$

where the coefficient of kinetic friction is given by $c = mL\gamma$, $\omega^2 = g/L$ and f is an arbitrary external force. Classify this differential equation by type, order and linearity.

1.1.2. Small Angle Approximation. There is no way to solve (6) generally, which is typical of higher order nonlinear equations. However, we can apply to so-called *small angle approximation*. To do this rewrite (6) replacing $\sin(\theta)$ with its Taylor series. If θ is less than one then the higher order terms decrease rapidly and can be discarded. The highest power kept is called the *order* of the approximation. Write down the ODE associated with first-order approximation to the sine function. What equation is this? Can it be solved?

1.1.3. Nonlinear Effects: Conservative Case. So, what can be done when we keep either some or all of the nonlinear terms? Well, not a lot generally. However, if we turn off the viscosity/dampening and any external forces then we expect that any energy we put into the pendulum stays in the pendulum for all time. That is, we expect that the system is conservative. Write down the undamped and unforced version of (6) and, using the energy method described in class, derive the conservation law associated with the ODE.

³ As we saw earlier in the semester, this assumption can be questioned and in more sophisticated fluid models we appeal to Lord Rayleigh's drag equation, which is quadratic in the velocity variable. See: http://en.wikipedia.org/wiki/Drag_equation

⁴Just as with friction, the assumption can be replaced in favor of a more precise model. Doing so essentially asks whether the spring is in the elastic regime, http://en.wikipedia.org/wiki/Elastic_limit, which is characterized by Hooke's law, http://en.wikipedia.org/wiki/Hooke's_law, where strain is directly proportional to stress. Outside of this limit or for springs in more exotic situations, Hook's law is not a good approximation and nonlinear effects must be taken into account.

1.2. **Undamped, Unforced Pendulum: Solutions.** It is possible to tease a bit of information out of the undamped, unforced pendulum equation,

$$(7) \quad \ddot{\theta} + \omega^2 \sin(\theta) = 0.$$

The starting point is its associated conservation law,

$$(8) \quad \frac{\dot{\theta}^2}{2} - \frac{\omega^2}{2} \cos(\theta) = E \in \mathbb{R},$$

which is a first-order separable ODE.

1.2.1. *Separation of Variables I: Elliptic Integrals.* Using separation of variables find $\theta(t)$.⁵

1.2.2. *Separation of Variables II: Small Angle Approximation.* Assume $\theta \ll 1$ and, using the quadratic approximation to cosine, find $\theta_2(t)$, which solves the second order approximation to the ODE.

1.2.3. *Separation of Variables III: Nonlinear Effects, Fail.* Assume $\theta^2 \ll 1$ and, using the quartic approximation to cosine, find $\theta_4(t)$, which solves the fourth order approximation to the ODE.⁶

Quote of LRC Circuits	
<p>Homer Simpson: A big mountain of sugar is too much for one man. I can see now why God portions it out in those little packets.</p>	
	The Simpsons: Lisa's Rival, S06E02 (1994)

2. OSCILLATIONS AND LRC CIRCUITS

For this problem we develop a correspondence between our mass-spring oscillations and circuits that have inductors, resistors and capacitors in series. Such a circuit is called an LRC circuit and will behave very much like a mass-spring system. For more information see:

- General Information: http://en.wikipedia.org/wiki/RLC_circuit
- Circuit Diagram: http://en.wikipedia.org/wiki/File:RLC_series_circuit.png
- LRC in Series: http://en.wikipedia.org/wiki/RLC_circuit#Series_RLC_circuit

2.1. **LRC Model.** In reference to the circuit diagram we mention that the current I , measured in amperes, is a function of time t . The resistance R (ohms), the capacitance C (farads), and the inductance L (henrys) are all positive and assumed to be known. The impressed voltage V (volts) is a given function of time. We will also need to understand the total charge Q (coulombs) on the capacitor at time t . The relation between charge Q and current I is $I = Q'$.

To arrive at the model equation we apply Kirchhoff's second law, which states that in a closed circuit the impressed voltage is equal to the sum of the voltage drops in the rest of the circuit. Another way of saying this is that the sum of potential voltage differences across a closed circuit must be zero.⁷ Now, we need only understand the voltage drops across the components. To this end we have:

- A resistor will slow the current according to: $I = v_R/R$
- A capacitor will store up charge and create a voltage difference according to: $v_C = Q/C$
- An inductor will create a voltage drop proportional to the time rate of change of the current according to: $v_L = LI'$

2.1.1. *Model Equation.* Using Kirchhoff's second law and the known information relating voltage drops to the circuit elements, derive the model equation for current I in the circuit as a function of time.

2.1.2. *Undriven LRC.* Assume that $V(t) = 0$ for all time and solve the differential equation $LI'' + RI' + C^{-1}I = V(t)$ for $I(t)$ and show that in the presence of resistance the current is a decaying function of time.

⁵ This case has been studied and can be managed through the so-called elliptic integrals. See: http://en.wikipedia.org/wiki/Elliptic_integral

⁶ Actually, you will just want to write down the integral. I can't solve it by hand. :(In fact, I tried a CAS system and it didn't much like it either. The moral here is that nonlinear terms are bad. Update! I'm still working on it and I think there is hope. Extra credit for those who want to journal some ideas about it. Ideas concluding in a solution will be pinned to my fridge.

⁷ This is similar to the idea that the line integral for a loop in a conservative field must be zero.

2.1.3. *Driven Vibrations.* Assume that $V(t) = F_0 \cos(\omega_0 t)$, $\omega_0 \in \mathbb{R}$ and solve for $I(t)$.

2.1.4. *Phase Angle.* The homogeneous solution is often called the transient solution since it decays in time. The particular solution is often called the steady-state solution since it persists. It is common to report the steady-state solution as one shifted trigonometric function containing a so-called phase angle. Specifically, show that $I_p(t)$ can be written as $I_p(t) = A \cos(\omega_0 t - \delta)$ where

$$(9) \quad A = \frac{F_0}{\Delta}$$

$$(10) \quad \cos(\delta) = \frac{L(\omega^2 - \omega_0^2)}{\Delta}$$

$$(11) \quad \sin(\delta) = \frac{R\omega}{\Delta}$$

$$(12) \quad \Delta = \sqrt{L^2(\omega^2 - \omega_0^2)^2 + R^2\omega^2}$$

$$(13) \quad \omega^2 = \frac{1}{LC}$$

This allows one to understand the steady-state response in terms of an amplitude A at the cost of a cosine shifted by δ , which is called the phase angle.⁸

2.1.5. *Steady-State Analysis.* What is A for $\omega_0 \rightarrow 0$? What is A for $\omega_0 \rightarrow \infty$?⁹

2.1.6. *Maximum Steady-State Amplitude.* Find the value of ω_0 for which the amplitude A is maximum. Find A at this value. What occurs to this amplitude when $R \rightarrow 0$?

3. LAPLACE TRANSFORMS

4. POWER SERIES AND QUANTUM MECHANICS

5. PHASE ANALYSIS OF SYSTEMS AND THE TRACE DETERMINANT PLANE

6. NONLINEAR SYSTEMS AND LINEARIZATION

⁸The idea is that we know the response is going to be trigonometric in nature and the δ is just an offset on where to begin the oscillations. The point is that we are most concerned about amplitude.

⁹This will show that for low frequency forcing the amplitude has a nonzero finite value and that for high frequency forcing the amplitude tends to zero. This now begs the question, is there a frequency between such that the amplitude achieves a maximum?