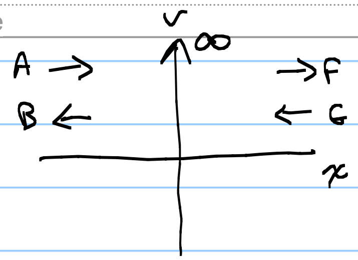
S - matrix revised

Note Title 2/24/2008

$\beta = \frac{m\alpha}{\hbar^2 k}$



we showed previously:

$$F + G = A + B$$
$$F - G = A(1+2i\beta) - B(1-2i\beta)$$

want B, F in terms of A, G

$$\boxed{\begin{array}{ccc} \text{out} & & \text{in} \\ \begin{bmatrix} B \\ F \end{bmatrix} = & \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} & \begin{bmatrix} A \\ G \end{bmatrix} \end{array}}$$

$$F - B = A - G$$
$$F + (1-2i\beta)B = (1+2i\beta)A + G$$

LHS RHS

$$\begin{bmatrix} -1 & 1 \\ 1-2i\beta & 1 \end{bmatrix}\begin{bmatrix} B \\ F \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ (1+2i\beta) & 1 \end{bmatrix}\begin{bmatrix} A \\ G \end{bmatrix}.$$

$$\Rightarrow \begin{bmatrix} B \\ F \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 1 \\ 1-2i\beta & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 & -1 \\ (1+2i\beta) & 1 \end{bmatrix}}\begin{bmatrix} A \\ G \end{bmatrix}$$

$$= \frac{1}{i+\beta}\begin{bmatrix} -\beta & i \\ i & -\beta \end{bmatrix} = S$$

$$|S_{22}|^2 = |S_{11}|^2 = \frac{\beta^2}{1+\beta^2} \qquad |S_{21}|^2 = |S_{12}|^2 = \frac{1}{1+\beta^2}$$

NB $\quad \text{Det}\ \frac{1}{i+\beta}\begin{bmatrix} -\beta & i \\ i & -\beta \end{bmatrix} = \left(\frac{1}{i+\beta}\right)^2 \cdot (\beta^2+1)$

$$\frac{\beta - i}{\beta + i} = \frac{(\beta+i)(\beta-i)}{(\beta+i)(\beta+i)}$$

$$S^{\dagger} = \frac{1}{-i+\beta}\begin{bmatrix} -\beta & -i \\ -i & \beta \end{bmatrix}$$

$$S^{\dagger}S = \frac{1}{1+\beta^2}\begin{bmatrix} -\beta & -i \\ -i & -\beta \end{bmatrix}\begin{bmatrix} -\beta & i \\ i & -\beta \end{bmatrix} = \begin{bmatrix} 1+\beta^2 & 0 \\ 0 & 1+\beta^2 \end{bmatrix}$$

$$= 1$$

The S matrix is unitary

$$\begin{bmatrix} S_{11}^* & S_{21}^* \\ S_{12}^* & S_{22}^* \end{bmatrix}\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} |S_{11}|^2+|S_{21}|^2 & S_{11}^*S_{12}+S_{21}^*S_{22} \\ S_{11}S_{12}^*+S_{22}^*S_{21} & |S_{12}|^2+|S_{22}|^2 \end{bmatrix}$$

unitarity $\Rightarrow$

$$|S_{11}|^2+|S_{21}|^2 = 1 \qquad R_e + T_e = 1$$
$$|S_{12}|^2+|S_{22}|^2 = 1 \qquad R_r + T_r = 1$$

$$S_{11}S_{12}^* = -S_{21}S_{22}^* \qquad R_eT_e = R_rT_r$$
$$S_{11}^*S_{12} = -S_{21}^*S_{22} \qquad (1-T_e)T_e = (1-T_r)T_r$$
$$1-T_e^2 = 1-T_r^2$$

$$\left.\begin{aligned} \alpha &= -\beta \\ \alpha^* &= -\beta^* \end{aligned}\right\} \text{same}$$

$$\Rightarrow T_e^2 = T_r^2$$

S-matrix gives outgoing (B, F)
in terms of incoming (A, G)

what about Right (A, B)
in terms of left (F, G) ?

$$\begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} M_{11} & {}_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

Eg. δ - potential

$F + G = A + B$

$F - G = A(1+2i\beta) - B(1-2i\beta)$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} F \\ G \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ (1+2i\beta) & -(1-2i\beta) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}$$

$$\begin{bmatrix} F \\ G \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}$$

$$\begin{bmatrix} F \\ G \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ (1+2i\beta) & -(1-2i\beta) \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1+i\beta & i\beta \\ -i\beta & 1-i\beta \end{bmatrix}}_{\text{transfer matrix}}\begin{bmatrix} A \\ B \end{bmatrix}$$

out $\begin{bmatrix} B \\ F \end{bmatrix} = S\begin{bmatrix} A \\ G \end{bmatrix}$ in

right $\begin{bmatrix} F \\ G \end{bmatrix} = M\begin{bmatrix} A \\ B \end{bmatrix}$ left

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For HW you will show

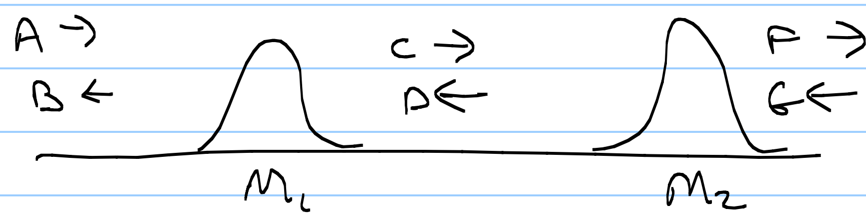
$$S = \frac{1}{M_{22}} \begin{pmatrix} -M_{21} & 1 \\ \det(M) & M_{12} \end{pmatrix}$$

$$M = \frac{1}{S_{12}} \begin{pmatrix} -\det(S) & S_{22} \\ -S_{11} & 1 \end{pmatrix}$$

So $R_L = |S_{11}|^2 = \left|\frac{M_{21}}{M_{22}}\right|^2 \quad T_L = |S_{21}|^2 = \left|\frac{\det(M)}{M_{22}}\right|^3$

$R_R = |S_{22}|^2 = \left|\frac{M_{12}}{M_{22}}\right|^2 \quad T_R = |S_{12}|^2 = \left|\frac{1}{M_{22}}\right|^2$

Now suppose you have 2 isolated potentials



$$\begin{pmatrix} F \\ G \end{pmatrix} = M_2 \begin{pmatrix} C \\ D \end{pmatrix} \qquad \begin{pmatrix} C \\ D \end{pmatrix} = M_1 \begin{bmatrix} A \\ B \end{bmatrix} \Longrightarrow$$

$$\begin{pmatrix} F \\ G \end{pmatrix} = M_2 \, M_1 \begin{bmatrix} A \\ B \end{bmatrix}$$