MATH348-Advanced Engineering Mathematics		**Homework: PDE Part I**
	CONSERVATIONS LAWS, SOLUTIONS TO PDE, READINGS	
Text: Chapter 12.1-12.2	Lecture Notes : N/A	Lecture Slides: N/A

Quote of Homework: PDE Part I
In life there an infinitely-many directions and each one is permitted.
CSM Professor Emeritus: John DeSanto - Mathematical Physics (2007)

1. Conservation Laws in One-Dimension

Recall that the conservation law encountered during the derivation of the heat equation was given by,

$$\frac{\partial u}{\partial t} = -\kappa \nabla \phi, \tag{1}$$

which reduces to

$$\frac{\partial u}{\partial t} = -\kappa \frac{\partial \phi}{\partial x},\ \kappa \in \mathbb{R} \tag{2}$$

in one-dimension of space.[1] In general, if the function $u = u(x,t)$ represents the density of a physical quantity then the function $\phi = \phi(x,t)$ represents its flux. If we assume the ϕ is proportional to the negative gradient of u then, from (2), we get the one-dimensional heat/diffusion equation.[2]

1.1. **Transport Equation.** Assume that ϕ is proportional to u to derive, from (2), the convection/transport equation, $u_t + cu_x = 0 \;\; c \in \mathbb{R}$.

1.2. **General Solution to the Transport Equation.** Show that $u(x,t) = f(x - ct)$ is a solution to this PDE.

1.3. **Diffusion-Transport Equation.** If both diffusion and convection are present in the physical system then the flux is given by, $\phi(x,t) = cu - du_x$, where $c, d \in \mathbb{R}^+$. Derive from, (2), the convection-diffusion equation $u_t + cu_x - du_{xx} = 0$.

1.4. **Convection-Diffusion-Decay.** If there is also energy/particle loss proportional to the amount present then we introduce to the convection-diffusion equation the term λu to get the convection-diffusion-decay equation,[3]

1.5. **General Importance of Heat/Diffusion Problems.** Given that,

$$u_t = Du_{xx} - cu_x - \lambda u. \tag{3}$$

Show that by assuming, $u(x,t) = w(x,t)e^{\alpha x - \beta t}$, (3) can be transformed into a heat equation on the new variable w where $\alpha = c/(2D)$ and $\beta = \lambda + c^2/(4D)$.[4]

2. Some Solutions to common PDE

Show that the following functions are solutions to their corresponding PDE's.

2.1. **Right and Left Travelling Wave Solutions.** $u(x,t) = f(x - ct) + g(x + ct)$ for the 1-D wave equation.

2.2. **Decaying Fourier Mode.** $u(x,t) = e^{-4\omega^2 t}\sin(\omega x)$ where $c = 2$ for the 1-D heat equation.

2.3. **Radius Reciprocation.** $u(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$ for the 3-D Laplace equation.

[1] When discussing heat transfer this is known as Fourier's Law of Cooling. In problems of steady-state linear diffusion this would be called Fick's First Law. In discussing electricity u could be charge density and q would be its flux.

[2] AKA Fick's Second Law associated with linear non-steady-state diffusion.

[3] The u_{xx} term models diffusion of energy/particles while u_x models convection, u models energy/particle loss/decay. The final term should not be surprising! Wasn't the appropriate model for radioactive/exponential decay $Y' = -\alpha^2 Y$?

[4] This shows that the general PDE (3) can be solved using heat equation techniques.

2.4. **Driving/Forcing Affects.** $u(x, y) = x^4 + y^4$ where $f(x, y) = 12(x^2 + y^2)$ for the 2-D Poisson equation.

Note: The PDE in question are,

- Laplace's equation : $\triangle u = 0$
- Poisson's equation : $\triangle u = f(x, y, z)$
- Heat/Diffusion Equation : $u_t = c^2 \triangle u$
- Wave Equation : $u_{tt} = c^2 \triangle u$

and can be found on page 563 of Kryszig. The following will outline some common notations. It is assumed all differential operators are being expressed in Cartesian coordinates.[5]

3. The Heat Equation

The essential idea of the second law of thermodynamics is that the thermodynamic quantities of closed systems tend to equilibrate over time. In our case the macroscopic stuff (heat energy) flows from areas of high density (temperature) to low density (temperature) and that in the long-term the densities will average-out until there is no longer a flow of stuff.

In this way the heat equation represents an ideal flow of a conserved quantity that obeys the second law of thermodynamics. So, though we frame this discussion in terms of heat it is usable for the diffusion of a conserved mass and due to its averaging nature appears in probability theory as well as some areas of financial mathematics. [6] The study of PDE is a broad mathematical topic rooted in natural phenomenon. For those concerned with the latter I offer the following readings to familiarize yourself with the fundamental concepts. Please read:

- The Introduction: `http://en.wikipedia.org/wiki/Heat_equation`
- The General Description: `http://en.wikipedia.org/wiki/Heat_equation#General_description`
- Subsection on Separation of Variables: `http://en.wikipedia.org/wiki/Heat_equation#Solving_the_heat_equation_using_Fourier_series`
- Subsection on Solutions via Green's Functions: `http://en.wikipedia.org/wiki/Heat_equation#Fundamental_solutions`.

3.1. **Equation Summary.** In your own words summarize the meaning of the heat equation. Be sure to address the physical interpretation of the solution to the heat equation. Also, list some applications you found interesting.

3.2. **Separation Summary.** In words describe the separation of variables solution technique. It turns out that the website solves the problem we just finished in class. However, they generalize this solution technique. In your words describe this generalization. That is, if you assume that the solution is energetically reasonable then where does your intuition take you?

3.3. **Green's Functions Summary.** Sometimes a Green's function is called a fundamental solution. What is the Green's function for the heat equation? What does this function look like for fixed t? What does this function do as t gets larger? What does it do when t gets smaller?

4. Wave Equations

We saw in a previous problem that if you assume that the flux is directly proportional to the density then the conservation law becomes the PDE $u_t + u_x = 0$, which has the property that it takes initial data and transports it, without deformation, through space. From this pure transport equation the one-dimensional wave equation can be derived.

In another problem the one-dimensional wave equation can be shown to have solutions which are the superposition of two purely transported waveforms. This is characteristically different than the dynamics predicted by the heat equation.[7] For this reason the wave equation is used as an ideal model of disturbances propagating in ideally elastic medium. As a model the wave equation is found in electricity and magnetism as well as acoustics and fluid mechanics. When the waves are not propagating then they are said to be standing and in this case the wave equation provides the model equation for vibrations in musical instruments.

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[5] Of course others have worked out the common coordinate systems, which requires some elbow grease and the multivariate chain rule. Those interested in the results can find them at Nabla in Cylindrical and Spherical

[6] See also:

- `http://en.wikipedia.org/wiki/Fokker-Planck_equation`
- `http://en.wikipedia.org/wiki/Diffusion_equation`
- `http://en.wikipedia.org/wiki/Black-Scholes`

[7] This pure transport is a feature of wave equations in one-dimension and does not simply generalize to higher dimensions. Nonetheless, it is used to help characterize and build intuition with waves, which are generally quite complicated.

Please read:

- Through the Introduction: http://en.wikipedia.org/wiki/Wave_equation
- Pure Transport for Scalar Wave Equations: http://en.wikipedia.org/wiki/Wave_equation#General_solution and http://en.wikipedia.org/wiki/Wave_equation#Spherical waves
- Boundary Value Problems: http://en.wikipedia.org/wiki/Wave_equation#Problems_with_boundaries
- Group Velocity Introduction: http://en.wikipedia.org/wiki/Group_velocity
- Phase Velocity Introduction: http://en.wikipedia.org/wiki/Phase_velocity
- Dispersion Introduction: http://en.wikipedia.org/wiki/Dispersion_(water_waves)
- Dispersion Introduction: http://en.wikipedia.org/wiki/Dispersion_(optics)

4.1. **Summary of Applications.** What are some of the applications of the wave equation that you found interesting? For these applications what is the modeled phenomenon and what does the solution to the PDE represent?

4.2. **Traveling Waves.** Traveling waves, which are discussed above, do exist to some extent in higher dimensions. When can this occur and what does such a wave model?

4.3. **Standing Waves.** Standing waves on squares and strings are represented in space by Fourier series. What changes when the geometry changes to a circle? What about a sphere?

4.4. **Dispersion of Waves.** What is the group velocity of a wave? What is the phase velocity of a wave? What is the dispersion of waves and where does this occur in nature?

5. Aspects of Nonlinearity

Everything we learn about in MATH348 is about linear mathematics. This is because linear mathematics is well-understood. However, just as a tangent line approximation is a poor approximation of the global features of the curve, linear models are a poor approximation to global features of complicated nonlinear phenomenon. Nonlinear mathematics is hard but that doesn't mean that we can't get a glimpse of the phenomenon associated with nonlinear mathematics.

Look over:

- List of Nonlinear PDE: http://en.wikipedia.org/wiki/List_of_nonlinear_partial_differential_equations#List_of_equations

Check out the pretty pictures:

- Wolfram Explorer: http://demonstrations.wolfram.com/NonlinearWaveEquationExplorer/
- Wolfram Explorer: http://demonstrations.wolfram.com/NonlinearWaveEquations/
- Sonic Boom: http://en.wikipedia.org/wiki/Sonic_boom

5.1. **Nonlinear Models.** Pick a couple of equations from the list of nonlinear PDE and explain what the PDE models and what the unknown function represents.