MATH 348 - Advanced Engineering Mathematics November 30, 2009

Homework 10, Fall 2009 **Due**:December 9th, 2009

PARTIAL DIFFERENTIAL EQUATIONS - HEAT AND WAVE EQUATIONS

Consider the one-dimensional heat equation,

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad , \tag{1}$$

$$x \in (0, L)\,, \qquad t \in (0, \infty)\,, \qquad c^2 = \frac{K}{\sigma\rho}. \tag{2}$$

Equations (1)-(2) model the time-evolution of the temperature, $u = u(x,t)$, of a heat conducting medium in one-dimension. The object, of length L, is assumed to have a homogeneous thermal conductivity K, specific heat σ, and linear density ρ. That is, $K, \sigma, \rho \in \mathbb{R}^+$.

1. Consider the one-dimensional heat equation, (1)-(2), with the boundary conditions[1],

$$u_x(0,t) = 0, \;\; u_x(L,t) = 0, \tag{3}$$

$$u(x,0) = f(x). \tag{4}$$

   (a) Assume that the solution to (1)-(2) is such that $u(x,t) = F(x)G(t)$ and use separation of variables to find the general solution to (1)-(2), which satisfies (3)-(4). [2]

   (b) Describe how the long term behavior of the general solution to (1)-(4) changes as the thermal conductivity, K, is increased while all other parameters are held constant. Also, describe how the solution changes when the linear density, ρ, is increased while all other parameters are held constant.

   Define,

$$f(x) = \begin{cases} \dfrac{2k}{L}x, & 0 < x \leq \frac{L}{2} \\ \frac{2k}{L}(L-x), & \frac{L}{2} < x < L \end{cases} \tag{5}$$

   and for the following questions we consider the solution, u, to the heat equation given by, (1)-(2), which satisfies the initial condition given by (5). [3]

   (c) For $L = 1$ and $k = 1$, find the particular solution to (1)-(2) with boundary conditions (3)-(4) for when the initial temperature profile of the medium is given by (5). Show that $\lim_{t\to\infty} u(x,t) = f_{avg} = 0.5$.[4]

2. Recall the 1-D conservation law encountered during the derivation of the heat equation.[5]

$$\frac{\partial u}{\partial t} + \frac{\partial \phi}{\partial x} = 0 \tag{6}$$

In general, if the function $u = u(x,t)$ represents the density of a physical quantity then the function $\phi = \phi(x,t)$ represents its flux. If we assume the ϕ is proportional to the negative gradient of u then, from (6), we get the one-dimensional heat/diffusion equation (1).[6]

[1] Here the boundary conditions correspond to perfect insulation of both endpoints

[2] An insulated bar is discussed in examples 4 and 5 on page 557. The boundary value problem manifesting from this PDE corresponds to problem two from homework nine.

[3] When solving the following problems it would be a good idea to go back through your notes and the homework looking for similar calculations.

[4] It is interesting here to note that though the initial condition f doesn't appear to satisfy the boundary conditions its periodic Fourier extension does. That is, if you draw the even periodic extension of the initial condition then you would see that the slope is not well defined at the end points. Remembering that the Fourier series averages the right and left hand limits of the periodic extension of the function f at the endpoints shows that the boundary conditions are, in fact, satisfied, since the derivative of an average is the average of derivatives.

[5] When discussing heat transfer this is known as Fourier's Law of Cooling. In problems of steady-state linear diffusion this would be called Fick's First Law. In discussing electromagnetism u could be charge density and ϕ would be its flux.

[6] AKA Fick's Second Law associated with linear non-steady-state diffusion.

(a) Assume that ϕ is proportional to u to derive, from (6), the convection/transport equation, $u_t + cu_x = 0$, where c is some proportionality constant.

(b) Given the initial condition $u(x, 0) = u_0(x)$ for the convection equation, show that $u(x, t) = u_0(x - ct)$ is a solution to this PDE.

(c) If both diffusion and convection are present in the physical system then the flux is given by, $\phi(x, t) = cu - du_x$, where $c, d \in \mathbb{R}^+$. Derive from, (6), the convection-diffusion equation $u_t + cu_x - du_{xx} = 0$.

(d) If there is also energy/particle loss proportional to the amount present then we introduce to the convection-diffusion equation the term λu to get the convection-diffusion-decay equation,[7]

$$u_t = Du_{xx} - cu_x - \lambda u. \tag{7}$$

Show that by assuming, $u(x, t) = w(x, t)e^{\alpha x - \beta t}$, (7) can be transformed into a heat equation on the new variable w where $\alpha = c/(2D)$ and $\beta = \lambda + c^2/(4D)$.[8]

Consider the one-dimensional wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad , \tag{8}$$

$$x \in (0, L)\,, \qquad t \in (0, \infty)\,, \qquad c^2 = \frac{T}{\rho}. \tag{9}$$

Equations (1)-(2) model the time-evolution of the displacement, $u = u(x, t)$, of an elastic medium in one-dimension. The object, of length L, is assumed to have a homogenous lateral tension T, and linear density ρ. That is, $T, \rho \in \mathbb{R}^+$.

3. Consider the one-dimensional wave equation, (8)-(9), with the boundary conditions[9],

$$u_x(0, t) = 0, u_x(L, t) = 0, \tag{10}$$

and initial conditions,

$$u(x, 0) = f(x), \tag{11}$$

$$u_t(x, 0) = g(x). \tag{12}$$

(a) Assume that the solution to (8)-(9) is such that $u(x, t) = F(x)G(t)$ and use separation of variables to find the general solution to (8)-(9), which satisfies (10)-(12). [10] [11]

(b) Let $L = 1$ and $k = 1$ and find the particular solution, which satisfies the initial displacement, $f(x)$, given by (5) and has zero initial velocity for all points on the object.

4. Show that by direct substitution that the function $u(x, t)$ given by,

$$u(x, t) = \frac{1}{2}\left[u_0(x - ct) + u_0(x + ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(y)dy, \tag{13}$$

5. Consider the non-homogenous one-dimensional wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + F(x, t) \quad , \tag{14}$$

$$x \in (0, L)\,, \qquad t \in (0, \infty)\,, \qquad c^2 = \frac{T}{\rho}. \tag{15}$$

[7] The u_{xx} term models diffusion of energy/particles while u_x models convection, u models energy/particle loss/decay.

[8] This shows that the general PDE (7) can be solved using heat equation techniques.

[9] These boundary conditions imply that the object must have zero curvature at each endpoint.

[10] It is important to notice that the solution to the spatial portion of the problem is the same as the heat problem of hw5 prob2.

[11] Remember that in this case we have nontrivial solutions for $k_0 = 0$. You should find that $G_0(t) = C_1 + C_2 t$.

with boundary conditions and initial conditions,

$$u(0,t) = u(L,t) = 0, \tag{16}$$

$$u(x,0) = u_t(x,0) = 0. \tag{17}$$

Letting $F(x,t) = A\sin(\omega t)$ gives the following Fourier Series Representation of the forcing function F ,

$$F(x,t) = \sum_{n=1}^{\infty} f_n(t)\sin\left(\frac{n\pi x}{L}\right), \tag{18}$$

where

$$f_n(t) = \frac{2A}{n\pi}(1-(-1)^n)\sin(\omega t). \tag{19}$$

(a) Show that substitution of (18)-(19) into (14) yields the ODE, [12]

$$G_n'' + \left(\frac{cn\pi}{L}\right)^2 G_n = \frac{2A}{n\pi}\,(1-(-1)^n)\sin(\omega t). \tag{20}$$

(b) The solution to (20) is given by,

$$G_n(t) = G_n^h(t) + G_n^p(t), \tag{21}$$

where $G_n^h(t) = B_n\cos\left(\frac{cn\pi}{L}t\right) + B_n^*\sin\left(\frac{cn\pi}{L}t\right)$ is the homogenous solution and $G_n^p(t)$ is the particular solution to (20). [13]

i. If $\omega \neq cn\pi/L$ then what would the choice for $G_n^p(t)$ be, assuming you were solving for $G_n^p(t)$ using the method of undetermined coefficients? DO NOT SOLVE FOR THESE COEFFICIENTS[14]

ii. If $\omega = cn\pi/L$ then what would the choice for $G_n^p(t)$ be, assuming you were solving for $G_n^p(t)$ using the method of undetermined coefficients? DO NOT SOLVE FOR THESE COEFFICIENTS [15]

iii. For the latter case what is $\lim_{t\to\infty} u(x,t)$ and what does this limit imply physically?[16]

[12] I think the key point here is to notice how the in-homogeneity of the PDE leads to an in-homogeneity in the ODE.

[13] This could be quickly determined by using problem one from homework nine. What's important here is that the homogeneous solution provides oscillations found in the unforced case and the in-homogeneous solution will produce another Fourier sine series, which captures the dynamics induced by $F(x,t)$.

[14] No second guessing. The table from problem one on homework nine outlines the choice. Don't let all the symbols fool you this all that we care about is the functional form $\sin(\omega t)$.

[15] Basically what is being said here is that by tuning the external force to be the natural frequency, associated with the time dynamics, of a single Fourier modes causes that mode to resonate and dominate the long-term dynamics of the system.

[16] Now would be a good time to think about problem (1c) on homework 9 and problem four from homework 7. If these problem don't seem to have a point then consider chapter four of your differential equations text.