

MATH 348 - SPRING 2008

## HOMEWORK 7

1) CONSIDER THE ONE-DIMENSIONAL HEAT EQUATION

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (1)$$

$$x \in (0, L) \quad t \in (0, \infty) \quad c^2 = \frac{k}{\rho} \quad (2)$$

$$u_x(0, t) = 0 \quad u(L, t) = 0 \quad (3)$$

$$u(x, 0) = f(x) \quad (4)$$

(a) USE SEPARATION OF VARIABLES TO FIND THE GENERAL SOLUTION.

$$u(x, t) = F(x)G(t)$$

$$\frac{\partial u}{\partial t} = F(x)G'(t) \quad \frac{\partial^2 u}{\partial x^2} = F''(x)G(t)$$

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \Rightarrow F(x)G'(t) = c^2 F''(x)G(t)$$

$$\Rightarrow \frac{F''(x)}{F(x)} = \frac{G'(t)}{c^2 G(t)} = -K$$

$$F''(x) + KF(x) = 0 \quad F'(0) = F(L) = 0$$

IF  $K < 0$ 

$$F(x) = c_1 \cosh(\sqrt{-K}x) + c_2 \sinh(\sqrt{-K}x)$$

$$F'(0) = c_2 \sqrt{-K} = 0 \Rightarrow c_2 = 0$$

$$F(L) = c_1 \cosh(\sqrt{-K}L) = 0 \Rightarrow c_1 = 0$$

NO NON-TRIVIAL SOLUTIONS

IF  $K = 0$ 

$$F(x) = c_1 + x c_2$$

$$F'(0) = c_2 = 0$$

$$F(L) = c_1 = 0 \Rightarrow c_1 = 0$$

NO NON-TRIVIAL SOLUTIONS

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IF  $K > 0$ 

$$F(x) = C_1 \cos(\sqrt{K}x) + C_2 \sin(\sqrt{K}x)$$

$$F'(0) = C_2 \sqrt{K} = 0 \Rightarrow C_2 = 0$$

$$F(L) = C_1 \cos(\sqrt{K}L) \Rightarrow \sqrt{K} = \frac{(2n-1)\pi}{2L}$$

$$F(x) = \sum_{n=1}^{\infty} C_n \cos\left(\frac{(2n-1)\pi}{2L} x\right)$$

$$G'(t) + c^2 K G(t) = 0$$

$$\int \frac{dG}{G} = \int -c^2 K t$$

$$\ln(G) = -c^2 K t + C$$

$$G(t) = C e^{-K c^2 t}$$

$$u(x, t) = F(x)G(t) = \sum_{n=1}^{\infty} C_n e^{-c^2 \left(\frac{(2n-1)\pi}{2L}\right)^2 t} \cos\left(\frac{(2n-1)\pi}{2L} x\right)$$

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} C_n \cos\left(\frac{(2n-1)\pi}{2L} x\right) \leftarrow \text{FOURIER SERIES}$$

$$C_n = \frac{1}{L} \int_0^L f(x) \cos\left(\frac{(2n-1)\pi}{2L} x\right) dx$$

$$u(x, t) = \sum_{n=1}^{\infty} C_n e^{-c^2 \left(\frac{(2n-1)\pi}{2L}\right)^2 t} \cos\left(\frac{(2n-1)\pi}{2L} x\right)$$

WHERE  $C_n = \frac{1}{L} \int_0^L f(x) \cos\left(\frac{(2n-1)\pi}{2L} x\right) dx$

(b) WHAT HAPPENS TO THE TEMPERATURE IN THE MEDIUM AS  $t$  GETS LARGE?

AS  $t \rightarrow \infty$ ,  $u(x, t) \rightarrow 0$

HEAT IS LEAVING THE SYSTEM

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2. CONSIDER THE ONE-DIMENSIONAL HEAT EQUATION (1), (2) WITH THE BOUNDARY CONDITIONS

$$u_x(0, t) = 0 \quad u_x(L, t) = 0 \quad (5)$$

$$u(x, 0) = f(x) \quad (6)$$

- (a) USE SEPARATION OF VARIABLES TO FIND THE GENERAL SOLUTION

$$u(x, t) = F(x)G(t)$$

$$\frac{\partial u}{\partial t} = F(x)G'(t) \quad \frac{\partial^2 u}{\partial x^2} = F''(x)G(t)$$

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \Rightarrow F(x)G'(t) = c^2 F''(x)G(t)$$

$$\frac{F''(x)}{F(x)} = \frac{G'(t)}{c^2 G(t)} = -K$$

$$F''(x) + KF(x) = 0 \quad F'(0) = 0 \quad F'(L) = 0$$

IF  $K < 0$ 

$$F(x) = c_1 \cosh(\sqrt{-K}x) + c_2 \sinh(\sqrt{-K}x)$$

$$F'(0) = c_2 \sqrt{-K} = 0 \Rightarrow c_2 = 0$$

$$F'(L) = c_1 \sqrt{-K} \sinh(\sqrt{-K}L) = 0 \Rightarrow c_1 = 0$$

NO NON-TRIVIAL SOLUTIONS

IF  $K = 0$ 

$$F(x) = c_1 + x c_2$$

$$F'(0) = c_2 = 0$$

$$F'(L) = c_2 = 0 \Rightarrow F(x) = c$$

IF  $K > 0$ 

$$F(x) = c_1 \cos(\sqrt{K}x) + c_2 \sin(\sqrt{K}x)$$

$$F'(0) = c_2 \sqrt{K} = 0 \Rightarrow c_2 = 0$$

$$F'(L) = -c_1 \sqrt{K} \sin(\sqrt{K}L) = 0 \Rightarrow \sqrt{K} = \frac{n\pi}{L}$$

$$\Rightarrow F(x) = \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi}{L}x\right)$$

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$$F(x) = C + \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi}{L}x\right)$$

$$G'(t) + c^2 k G(t) = 0$$

$$\int \frac{dG}{G} = \int -c^2 k dt$$

$$\ln G = -c^2 k t + C$$

$$G(t) = C e^{-c^2 k t}$$

$$u(x,t) = C_0 + \sum_{n=1}^{\infty} C_n e^{-c\left(\frac{n\pi}{L}\right)^2 t} \cos\left(\frac{n\pi}{L}x\right)$$

↑  
SOLUTION FOR  
 $k=0$

↑  
SOLUTION FOR  $k>0$

$$u(x,0) = f(x) = C_0 + \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi}{L}x\right) \quad \leftarrow \text{FOURIER SERIES}$$

$$u(x,t) = C_0 + \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi}{L}x\right)$$

$$\text{WHERE } C_0 = \frac{1}{L} \int_0^L f(x) dx \quad C_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi}{L}x\right) dx$$

(b) DESCRIBE THE LONG TERM BEHAVIOR AS  $k$  IS INCREASED AND AS  $\rho$  IS INCREASED.

IF  $k$  (THERMAL CONDUCTIVITY) IS INCREASED, THE TEMPORAL SOLUTION DECAYS FASTER AND THE SYSTEM REACHES EQUILIBRIUM SOONER.

IF  $\rho$  (DENSITY) IS INCREASED, THE TEMPORAL SOLUTION DECAYS SLOWER AND THE SYSTEM TAKES LONGER TO REACH EQUILIBRIUM.

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$$f(x) = \begin{cases} x & 0 < x < L \\ -x + 2L & L < x < 2L \end{cases} \quad (*)$$

(\*) SUPPOSE THE BOUNDARY CONDITIONS ARE

$$u(0, t) = u(2L, t) = 0$$

DETERMINE THE UNIQUE SOLUTION. WHAT IS  $\lim_{t \rightarrow \infty} u(x, t)$ ?

$$u(x, t) = F(x)G(t)$$

$$\frac{\partial u}{\partial t} = F(x)G'(t) \quad \frac{\partial^2 u}{\partial x^2} = F''(x)G(t)$$

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \Rightarrow F(x)G'(t) = c^2 F''(x)G(t)$$

$$\frac{F''(x)}{F(x)} = \frac{G'(t)}{c^2 G(t)} = -K$$

$$F''(x) + KF(x) = 0 \quad F(0) = F(2L) = 0$$

IF  $K < 0$ 

$$F(x) = c_1 \cosh(\sqrt{-K}x) + c_2 \sinh(\sqrt{-K}x)$$

$$F(0) = c_1 = 0$$

$$F(2L) = c_2 \sinh(\sqrt{-K}2L) = 0 \Rightarrow c_2 = 0$$

NO NON-TRIVIAL SOLUTIONS

IF  $K = 0$ 

$$F(x) = c_1 + c_2 x$$

$$F(0) = c_1 = 0$$

$$F(2L) = c_2 2L = 0 \Rightarrow c_2 = 0$$

NO NON-TRIVIAL SOLUTIONS

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IF  $K > 0$ 

$$F(x) = C_1 \cos(\sqrt{K}x) + C_2 \sin(\sqrt{K}x)$$

$$F(0) = C_1 = 0$$

$$F(2L) = C_2 \sin(\sqrt{K}2L) = 0 \Rightarrow \sqrt{K} = \frac{n\pi}{2L}$$

$$F(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{2L}x\right)$$

$$G'(t) + c^2 K G(t) = 0$$

$$\int \frac{dG}{G} = c^2 K dt$$

$$\ln G = -c^2 K t + C$$

$$G(t) = C e^{-c^2 K t}$$

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-c^2 \left(\frac{n\pi}{2L}\right)^2 t} \sin\left(\frac{n\pi}{2L}x\right)$$

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{2L}x\right) = f(x) = \begin{cases} x & 0 < x \leq L \\ -x + 2L & L < x < 2L \end{cases}$$

$$C_n = \frac{2}{2L} \int_0^{2L} f(x) \sin\left(\frac{n\pi}{2L}x\right) dx$$

$$= \frac{1}{L} \left[ \int_0^L x \sin\left(\frac{n\pi}{2L}x\right) dx + \int_L^{2L} (-x+2L) \sin\left(\frac{n\pi}{2L}x\right) dx \right]$$

$$= \frac{1}{L} \left[ \left. -\frac{2Lx}{n\pi} \cos\left(\frac{n\pi}{2L}x\right) + \frac{2L^2}{n^2\pi^2} \sin\left(\frac{n\pi}{2L}x\right) \right|_0^L + \left. \frac{2Lx}{n\pi} \cos\left(\frac{n\pi}{2L}x\right) - \frac{2L^2}{n^2\pi^2} \sin\left(\frac{n\pi}{2L}x\right) - \frac{4L^2}{n\pi} \cos\left(\frac{n\pi}{2L}x\right) \right|_L^{2L} \right]$$

$$= \frac{1}{L} \left[ -\frac{2L^2}{n\pi} \cos\left(\frac{n\pi}{2}\right) + \frac{4L^2}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) + \frac{4L^2}{n\pi} \cos(n\pi) - \frac{4L^2}{n\pi} \cos(n\pi) - \frac{2L^2}{n\pi} \cos\left(\frac{n\pi}{2}\right) + \frac{4L^2}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) + \frac{4L^2}{n\pi} \cos\left(\frac{n\pi}{2}\right) \right]$$

$$= \frac{8L \sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2}$$

$$u(x,t) = \sum_{n=1}^{\infty} \left( \frac{8L \sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2} \right) e^{-c^2 \left(\frac{n\pi}{2L}\right)^2 t} \sin\left(\frac{n\pi}{2L}x\right)$$

$$\lim_{t \rightarrow \infty} u(x,t) = 0$$

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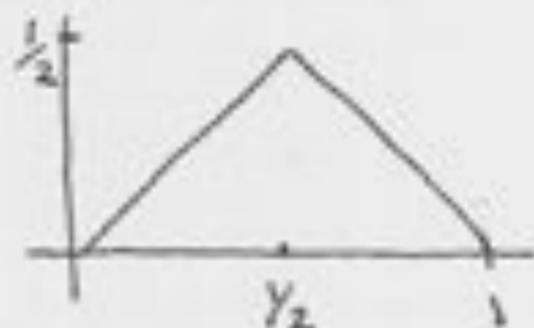
(6) FOR  $L=2L=1$ , FIND THE SOLUTION TO (1)-(2) WITH INITIAL CONDITION (7) AND BOUNDARY CONDITIONS (5)-(6). SHOW THAT  $\lim_{t \rightarrow \infty} u(x,t) = f_{\text{AVE}} = .25$

FROM (2) WE HAVE THAT

$$u(x,t) = C_0 + \sum_{n=1}^{\infty} \frac{e^{-c^2 \left(\frac{n\pi}{L}\right)^2 t}}{2Ln} \cos\left(\frac{n\pi}{L}x\right)$$

USING  $L=2L=1$ , (7) BECOMES

$$f(x) = \begin{cases} x & 0 \leq x \leq \frac{1}{2} \\ -x+1 & \frac{1}{2} < x < 1 \end{cases}$$



$$C_0 = \int_0^1 f(x) dx = \int_0^{1/2} x dx + \int_{1/2}^1 (-x+1) dx$$

$$= \frac{1}{2}x^2 \Big|_0^{1/2} + \left(-\frac{1}{2}x^2 + x\right) \Big|_{1/2}^1 = \frac{1}{4}$$

$$C_n = 2 \int_0^1 f(x) \cos(n\pi x) dx = 2 \left[ \int_0^{1/2} x \cos(n\pi x) dx + \int_{1/2}^1 (-x+1) \cos(n\pi x) dx \right]$$

$$= 2 \left[ \frac{x}{n\pi} \sin(n\pi x) + \frac{1}{n^2\pi^2} \cos(n\pi x) \Big|_0^{1/2} + \frac{-x}{n\pi} \sin(n\pi x) - \frac{1}{n^2\pi^2} \cos(n\pi x) + \frac{1}{n\pi} \sin(n\pi x) \Big|_{1/2}^1 \right]$$

$$= 2 \left[ \frac{1}{2n\pi} \sin\left(\frac{n\pi}{2}\right) + \frac{1}{n^2\pi^2} \cos\left(\frac{n\pi}{2}\right) - \frac{1}{n^2\pi^2} - \frac{1}{n^2\pi^2} \cos(n\pi) + \frac{1}{2n\pi} \sin\left(\frac{n\pi}{2}\right) + \frac{1}{n^2\pi^2} \cos\left(\frac{n\pi}{2}\right) - \frac{1}{n\pi} \sin\left(\frac{n\pi}{2}\right) \right]$$

$$= \frac{4}{n^2\pi^2} \left[ 2 \cos\left(\frac{n\pi}{2}\right) - 1 - \cos(n\pi) \right] = \frac{2(-1)^n - 2}{n^2\pi^2}$$

$$u(x,t) = \frac{1}{4} + \sum_{n=1}^{\infty} \left( \frac{2(-1)^n - 2}{n^2\pi^2} \right) e^{-c^2(2n\pi)^2 t} \cos(2n\pi x)$$

$$\lim_{t \rightarrow \infty} u(x,t) = C_0 = \frac{1}{L} \int_0^L f(x) dx = f_{\text{AVE}} = .25$$

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4 RECALL THE 1-D CONSERVATION LAW

$$\frac{\partial u}{\partial t} = -k \frac{\partial \phi}{\partial x} \quad (8)$$

(a) ASSUME THAT  $\phi$  IS PROPORTIONAL TO  $u$ , TO DERIVE THE CONVECTION/TRANSPORT EQUATION  $u_t + cu_x = 0$

$$\phi = \alpha u$$

$$\frac{\partial \phi}{\partial x} = \alpha \frac{\partial u}{\partial x}$$

$$\frac{\partial u}{\partial t} = -k \frac{\partial \phi}{\partial x} \Rightarrow \frac{\partial u}{\partial t} = -\alpha k \frac{\partial u}{\partial x} \Rightarrow u_t + cu_x = 0$$

(b) GIVEN THAT  $u(x, 0) = u_0(x)$ , SHOW THAT  $u(x, t) = u_0(x - ct)$  IS A SOLUTION

$$u(x, t) = u_0(x - ct)$$

$$u_t = -c u'_0 \quad u_x = u'_0$$

$$u_t + cu_x = -c u'_0 + c u'_0 = 0$$

(c) IF  $\phi(x, t) = cu - d u_{xx}$ , DERIVE FROM (8) THE CONVECTION-DIFFUSION EQUATION  $u_t + cu_x - d u_{xx} = 0$

$$\frac{\partial \phi}{\partial x} = c \frac{\partial u}{\partial x} - d \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u}{\partial t} = -k \frac{\partial \phi}{\partial x} \Rightarrow \frac{\partial u}{\partial t} = -c \frac{\partial u}{\partial x} + d \frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow u_t + cu_x - d u_{xx} = 0$$

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$$(d) \quad u_t = D u_{xx} - c u_x - \lambda u \quad (*)$$

ASSUME THAT  $u(x,t) = w(x,t) e^{\alpha x - \beta t}$  AND SHOW THAT  
 (a) CAN BE TRANSFORMED INTO A HEAT EQUATION ON  
 THE VARIABLE  $w$  WHERE  $\alpha = c/(2D)$  AND  $\beta = \lambda + c^2/(4D)$

$$u_t = w_t e^{\alpha x - \beta t} + w \beta e^{\alpha x - \beta t}$$

$$u_x = w_x e^{\alpha x - \beta t} + w \alpha e^{\alpha x - \beta t}$$

$$u_{xx} = w_{xx} e^{\alpha x - \beta t} + 2w_x \alpha e^{\alpha x - \beta t} + w \alpha^2 e^{\alpha x - \beta t}$$

$$u_t = D u_{xx} - c u_x - \lambda u$$

$$w_t e^{\alpha x - \beta t} + w \beta e^{\alpha x - \beta t} = D w_{xx} e^{\alpha x - \beta t} + D 2w_x \alpha e^{\alpha x - \beta t} + D w \alpha^2 e^{\alpha x - \beta t} - c w_x e^{\alpha x - \beta t} - c w \alpha e^{\alpha x - \beta t} - \lambda w e^{\alpha x - \beta t}$$

$$\Rightarrow w_t - \beta w = D w_{xx} + 2D \alpha w_x + D w \alpha^2 - c w_x - c \alpha w - \lambda w$$

$$w_t = D w_{xx} + (2D \alpha - c) w_x + (\beta - c \alpha + D \alpha^2 - \lambda) w$$

$$w_t = D w_{xx} + \left( 2D \left( \frac{c}{2D} - c \right) \right) w_x + \left( \lambda + \frac{c^2}{4D} - \frac{c^2}{2D} + \frac{D c^2}{4D^2} - \lambda \right) w$$

$$w_t = D w_{xx} \quad \leftarrow \text{HEAT EQUATION ON VARIABLE } w$$

5 (a) DEFINE CONVECTION, CONDUCTION AND RADIATION.

CONDUCTION: HEAT TRANSFER BY DIRECT PHYSICAL CONTACT

CONVECTION: HEAT TRANSFER BY FLUID CIRCULATION

CONVECTION: TRANSFER OF HEAT THROUGH ELECTROMAGNETIC RAYS

(b) WHY MUST HEAT TRANSFER OCCUR ~~BEFORE~~ FROM HOT TO COLD BODIES?

THE 2ND LAW OF THERMODYNAMICS STATES THAT ENTROPY CANNOT BE REVERSED WITHOUT DOING WORK

(c) GIVE EXAMPLES OF 3 DISCIPLINES EMPLOYING HEAT TRANSFER  
 CHEMICAL, MECHANICAL AND AUTOMOTIVE ENGINEERING