MATH348: SPRING 2012 - HOMEWORK 3

HALF-RANGE EXPANSIONS, COMPLEX FOURIER SERIES, PERIODIC FORCING

Lying on the floor, lying on the floor! I've come undone.

Date: February 12, 2012.

ABSTRACT. A *real* Fourier series is the linear combination of orthogonal functions from the set

$$S_1 = \{1, \cos(\omega_1 x), \sin(\omega_1 x, \cos(\omega_2 x), \sin(\omega_2 x), \cos(\omega_3 x), \sin(\omega_3 x), \dots\} \tag{1}$$

where the angular frequencies are counting multiples of the base frequency π/L. That is, $\omega_n = n\pi/L$ for $n = 1, 2, 3, \dots$. Since the elements in (1) all share a $2L-$period, their linear combination

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\omega_n x) + b_n \sin(\omega_n x) \tag{2}$$

also shares this period. Essentially what we have is a method of writing the $2L-$periodic function $f(x)$ in terms of the sum of the well-known $2L-$periodic circular trigonometric functions. One way to think about (2) is that $f(x)$ is the interference pattern, created by the constructive and destructive interference of the sinusoid waves, whose constituent wave amplitudes are given by the Fourier coefficients,

(3)

$$a_0 = \frac{1}{2L}\int_a^b f(x)dx, \quad a_n = \frac{1}{L}\int_a^b f(x)\cos(\omega_n x)dx, \quad b_n = \frac{1}{L}\int_a^b f(x)\sin(\omega_n x)dx,$$

where $b - a = 2L$. Yet another way to think about a Fourier series is that (3) takes in the data/graph $f(x)$, on the interval $I = (a, b)$, and then (2) repeats the original data every $2L-$units to the right and left of the original interval I. Also, written in this way, from the symmetry of the functions in (2) we can see that if $f(x)$ is even(odd) then the $b_n(a_n)$ coefficients vanish. Together, these notions gives us a way to *extend* Fourier series to functions that aren't necessarily periodic. To motivate this idea let's consider a function $f(x)$ representing a physical quantity of an object that is $l-$units long, our idea goes like this:

1. First we must place the object in a coordinate system and it makes the most sense to let $f(x)$ be defined for $x \in (0, l)$.
2. Though $f(x)$ is not periodic we can find a periodic function $f^*(x)$ such that $f(x) = f^*(x)$ for $x \in (0, l)$. We call f^* the *periodic extension* of f and it is only possible to define this function because f is not specified outside of $(0, l)$. That is, the physical object does not exist outside of this interval and consequently we can mathematically use this space to define f^*.
3. The periodic extension of a function is not unique. Since we are using the rest of the space to define f^* it makes sense to introduce a symmetry first so that the associated Fourier series simplifies. When both a symmetry and a periodic extension are introduced a *half-range expansion* is the result.

So, the Fourier series is a tool that is applicable to any reasonable function defined on a finite portion of space. In the following problems we continue our work with Fourier series in the following ways:

- P1-P2. It is important to remember that the Fourier series can be formulated on any period with finite length. With this in mind, you should always choose a domain of integration that makes the integrals as simple as possible.
- P3. We will need this triangle for the rest of the semester. In class we will construct the Fourier sine half-range expansion of f, it's your job to construct its even half-range expansion.
- P4. Based on our discussions in class, it should not be a surprise that the elements of $S_2 = \left\{\dots, e^{-\omega_2 x}, e^{-\omega_1 x}, 1, e^{\omega_1 x}, e^{\omega_2 x}, \dots\right\}$ can be used to construct a Fourier series. This problem is about the so-called *complex* Fourier series and its equivalence with the real Fourier series.
- P5. In differential equations you were exposed to the forced simple harmonic oscillator,

  $$my'' + ky = \cos(\omega t), \quad m, k, \omega \in \mathbb{R}^+. \tag{4}$$

  Specifically, the system will *beat* for $\omega \approx \sqrt{k/m}$ and *resonate* for $\omega = \sqrt{k/m}$. That said, it is unlikely that system will be forced in an obvious sinusoidal way but likely it is forced periodically. This problem is about resonance that may be induced by periodic forcing.

1. Fourier Series : Nonstandard Domain

Let $f(x) = x^2$ for $x \in (0, 2\pi)$ be such that $f(x+2\pi) = f(x)$.

1.1. **Graphing.** Sketch f on $(-4\pi, 4\pi)$.

1.2. **Symmetry.** Is the function even, odd or neither?

1.3. **Integrations.** Determine the Fourier coefficients a_0, a_n, b_n of f.

1.4. **Truncation.** Using `http://www.tutor-homework.com/grapher.html`, or any other graphing tool, graph the first five terms of your Fourier Series Representation of f.

2. Fourier Series : Nonstandard Period

Let $f(x) = \begin{cases} 0, & -2 < x < 0 \\ x, & 0 < x < 2 \end{cases}$ be such that $f(x+4) = f(x)$.

2.1. **Graphing.** Sketch f on $(-4, 4)$.

2.2. **Symmetry.** Is the function even, odd or neither?

2.3. **Integrations.** Determine the Fourier coefficients a_0, a_n, b_n of f.

2.4. **Truncation.** Using `http://www.tutor-homework.com/grapher.html`, or any other graphing tool, graph the first five terms of your Fourier Series Representation of f.

3. Fourier Series : Periodic Extension

Let $f(x) = \begin{cases} \dfrac{2k}{L}x, & 0 < x \leq \frac{L}{2} \\ \frac{2k}{L}(L-x), & \frac{L}{2} < x < L \end{cases}$.

3.1. **Graphing - I.** Sketch a graph f on $[-2L, 2L]$.

3.2. **Graphing - II.** Sketch a graph f^*, the even periodic extension of f, on $[-2L, 2L]$.

3.3. **Fourier Series.** Calculate the Fourier cosine series for the half-range expansion of f.

3.4. **Truncation.** Using `http://www.tutor-homework.com/grapher.html`, or any other graphing tool, graph the first five terms of your Fourier Series Representation of f where $k = L = 1$.

4. Complex Fourier Series

4.1. **Orthogonality Results.** Show that $\left\langle e^{inx}, e^{imx} \right\rangle = 2\pi\delta_{nm}$ where $n, m \in \mathbb{Z}$, where $\langle f, g \rangle = \displaystyle\int_{-\pi}^{\pi} f(x)\overline{g(x)}dx$.

4.2. **Fourier Coefficients.** Using the previous orthogonality relation find the Fourier coefficients, c_n, for the complex Fourier series, $f(x) = \displaystyle\sum_{n=-\infty}^{\infty} c_n e^{inx}$.

4.3. **Complex Fourier Series Representation.** Find the complex Fourier coefficients for $f(x) = x^2,\ -\pi < x < \pi,\ f(x+2\pi) = f(x)$.

4.4. **Conversion to Real Fourier Series.** Using the complex Fourier series representation of f recover its associated real Fourier series.

5. Periodic Forcing of Simple Harmonic Oscillators

Consider the ODE, which is commonly used to model forced simple harmonic oscillation,

$$y'' + 9y = f(t), \tag{5}$$

$$f(t) = |t|,\ -\pi \leq t < \pi,\ f(t+2\pi) = f(t). \tag{6}$$

Since the forcing function (6) is a periodic function we can study (5) by expressing $f(t)$ as a Fourier series. [1] [2]

5.1. **Fourier Series Representation.** Express $f(t)$ as a real Fourier series.

5.2. **Method of Undetermined Coefficients.** The solution to the homogeneous problem associated with (5) is $y_h(t) = c_1 \cos(3t) + c_2 \sin(3t),\quad c_1, c_2 \in \mathbb{R}$. Knowing this, if you were to use the method of undetermined coefficients[3] then what would your choice for the particular solution, $y_p(t)$? DO NOT SOLVE FOR THE UNKNOWN CONSTANTS

5.3. **Resonant Modes.** What is the particular solution associated with the third Fourier mode of the forcing function?[4]

5.4. **Structural Changes.** What is the long term behavior of the solution to (5) subject to (6)? What if the ODE had the form $y'' + 4y = f(t)$?

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[1] The advantage of expressing $f(t)$ as a Fourier series is its validity for any time t. An alternative approach have been to construct a solution over each interval $n\pi < t < (n+1)\pi$ and then piece together the final solution assuming that the solution and its first derivative is continuous at each $t = n\pi$.

[2] It is worth noting that this concepts are used by structural engineers, a sub-disciple of civil engineering, to study the effects of periodic forcing on buildings and bridges. In fact, this problem originate from a textbook on structural engineering.

[3] This is also known as the method of the 'lucky guess' in your differential equations text.

[4] Each term in a Fourier series is called a mode. The first mode is sometimes called the fundamental mode. The rest of the modes, called *harmonics* in acoustics, are just referenced by number. The third Fourier mode would be the third term of Fourier summation