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$$Y(z) = \frac{1}{1-z} = (1-z)^{-1}$$

$$Y' = \frac{+1}{(1-z)^2}$$

$$Y'' = \frac{2}{(1-z)^3}$$

$$\Rightarrow \quad Y'' - \frac{2}{(1-z)^2} Y = \frac{2}{(1-z)^3} - \frac{2}{(1-z)^2}\frac{1}{(1-z)} = 0$$

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Hydrogen atom. Radial part of Schrödinger Eqn. leads to

$$\frac{d^2v}{dx^2} = \left(\epsilon - \frac{2}{x}\right) v(x)$$

For large x this equation reduces to

$$\boxed{\frac{d^2v}{dx^2} = \epsilon v(x)}$$ important example of putting information into problem

So as $\underset{x\to\infty}{}$ $v(x) \to e^{\pm\sqrt{\epsilon}\,x}$

$$\Rightarrow \quad v(x) \to e^{-\sqrt{\epsilon}\,x}$$

only physically reasonable solution

So, Let's look for solutions of the form

$$v(x) = e^{-\alpha x} f(x) \qquad \alpha^2 = \epsilon$$

$$\frac{d^2 v}{dx^2} = \left(\epsilon - \frac{2}{x}\right) v(x)$$

•) $v'(x) = -\alpha e^{-\alpha x} f(x) + e^{-\alpha x} f'$

$$v'' = \alpha^2 e^{-\alpha x} f(x) - \alpha e^{-\alpha x} f' - \alpha e^{-\alpha x} f' + e^{-\alpha x} f''$$

$$= (\alpha^2 f - 2\alpha f' + f'') e^{-\alpha x}$$

$$= \left(\alpha^2 - \frac{2}{x}\right) e^{-\alpha x} f(x)$$

$$\Rightarrow \boxed{f'' - 2\alpha f' + \frac{2}{x} f(x) = 0}$$

Series Solution

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

Recall: $$f' = \sum_{k=0}^{\infty} (k+1) a_{k+1} x^k$$

$$f'' = \sum_{k=0}^{\infty} (k+1)(k+2) a_{k+2} x^k$$

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$$\sum_0^\infty (k+1)(k+2)a_{k+2}x^k - 2\alpha\sum_0^\infty (k+1)a_{k+1}x^k$$
$$+\frac{2}{x}\sum_0^\infty a_k x^k = 0$$

multiply by x

$\Rightarrow$

$$\sum_0^\infty (k+1)(k+2)a_{k+2}x^{k+1} - 2\alpha\sum_0^\infty (k+1)a_{k+1}x^{k+1}$$
$$+2\sum_0^\infty a_k x^k = 0$$

$k+1 \to N$

shift $$\sum_{N=0}^\infty N(N+1)\underbrace{a_{N+1}}_{a_{N+1}}x^N - 2\alpha\sum_{N=0}^\infty N a_N x^N + 2\sum_{N=0}^\infty \underbrace{a_N}_{a_N} x^N$$

$$\Rightarrow \sum_{N=0}^\infty \Big[\underbrace{N(N+1)a_{N+1} - 2\alpha N a_N + 2a_N}_{=0}\Big]x^N = 0$$

$$\boxed{a_{N+1} = \frac{2(\alpha N-1)}{N(N+1)}a_N}$$

1-term recursion. if we fix a_1 $\left[a_0 = 0 \quad \frac{1}{N}\right]$

so

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$$a_{N+1} = \frac{2(\alpha N - 1)}{N(N+1)} a_N$$

$$a_2 = \frac{2(\alpha - 1)}{2 \cdot 1} a_1$$

$$a_3 = \frac{2(2\alpha - 1)}{3 \cdot 2} \times \frac{2(\alpha - 1)}{2 \cdot 1} a_1$$

$$a_4 = \frac{2(3\alpha - 1)}{4 \cdot 3} \cdot \frac{2(2\alpha - 1)}{3 \cdot 2} \frac{2(\alpha - 1)}{2 \cdot 1} a_1$$

$$= \frac{2^3 \alpha^3 (3 - 1/\alpha)(2 - 1/\alpha)(1 - 1/\alpha)}{4! \quad 3!} a_1$$

$$a_{N+1} \cong \frac{2^N \alpha^N N!}{(N+1)! \; N!} \approx \frac{2^N \alpha^N}{N!} a_1$$

$$\Rightarrow \quad f(x) = \sum_{N=0}^{\infty} \frac{(2\alpha)^N}{N!} x^N a_1$$

$$\sim a_1 e^{2\alpha x}$$

$$\Rightarrow \quad v(x) = a_1 e^{2\alpha x} e^{-\alpha x} \underbrace{= a_1 e^{\alpha x}}_{\text{blows up!}}$$

what gives?

the only way for ~~[illegible]~~ $v(x)$ to remain finite is for the power series to terminate:

$$a_{N+1} = \frac{2(\alpha N - 1)}{N(N+1)} a_N$$

Suppose for some N' $\alpha N' = 1$

then $a_{N'+1} = 0$ and all subsequent a_N are zero.

So, the possible values of $\alpha^2 = \epsilon$ are labeled by ~~[illegible]~~ an integer

$$\boxed{\epsilon_N = \frac{1}{N^2}}$$

should look familiar for H atom

Legendre polynomials

$$(1-x^2)\,y'' - 2xy' + \ell(\ell+1)\,y = 0$$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} (n+1)\,a_{n+1}\,x^n$$

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1)\,a_{n+2}\,\cancel{x}^n$$

$$(1-x^2)\left[\sum_{n=0}^{\infty} (n+2)(n+1)\,a_{n+2}\,x^n\right] - 2x\left[\sum_{n=0}^{\infty} (n+1)\,a_{n+1}\,x^n\right] + \ell(\ell+1)\sum_{n=0}^{\infty} a_n x^n$$

$\Rightarrow$

$$\sum (n+2)(n+1)\,a_{n+2}\,x^n - \sum (n+2)(n+1)\,a_{n+2}\,x^{n+2}$$

$$-2\sum (n+1)\,a_{n+1}\,x^{n+1} + \ell(\ell+1)\sum a_n x^n$$

shift $\Rightarrow$

$$\sum (n+1)(n+2)\,a_{n+2}\,x^n - \sum n(n-1)\,a_n\,x^n$$

$$-2\sum n\,a_n\,x^n + \ell(\ell+1)\sum a_n x^n$$

$$\sum \left[(n+1)(n+2)\,a_{n+2} - n(n-1)\,a_n - 2n\,a_n + \ell(\ell+1)\,a_n\right] x^n$$

$$= 0$$

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$$(N+1)(N+2)\,a_{N+2} - \left[N(N-1) + 2N - \ell(\ell+1) \right] a_N = 0$$

$$a_{N+2} = \frac{N(N-1) + 2N - \ell(\ell+1)}{(N+1)\,(N+2)}$$

$$\boxed{a_{N+2} = \frac{N(N+1) - \ell(\ell+1)}{(N+1)\,(N+2)}\, a_N}$$

E.g. $a_0 = 1 \qquad a_1 = 0$

$$a_2 = \frac{-\ell(\ell+1)}{2}$$

$$a_4 = \frac{2(2+1) - \ell(\ell+1)}{4 \cdot 3} \cdot \frac{-\ell(\ell+1)}{2}$$

$$Y(x) = 1 - \frac{\ell(\ell+1)}{2}\, x^2$$

$$+ \frac{\ell^2(\ell+1)^2 - 6\,\ell(\ell+1)}{4!}\, x^4$$