MATH 225 - June 12, 2008
Exam IV - 50 minutes - 50 Points

NAME:

**In order to receive full credit, SHOW ALL YOUR WORK. Full credit will be given only if all reasoning and work is provided. When applicable, please enclose your final answers in boxes.**

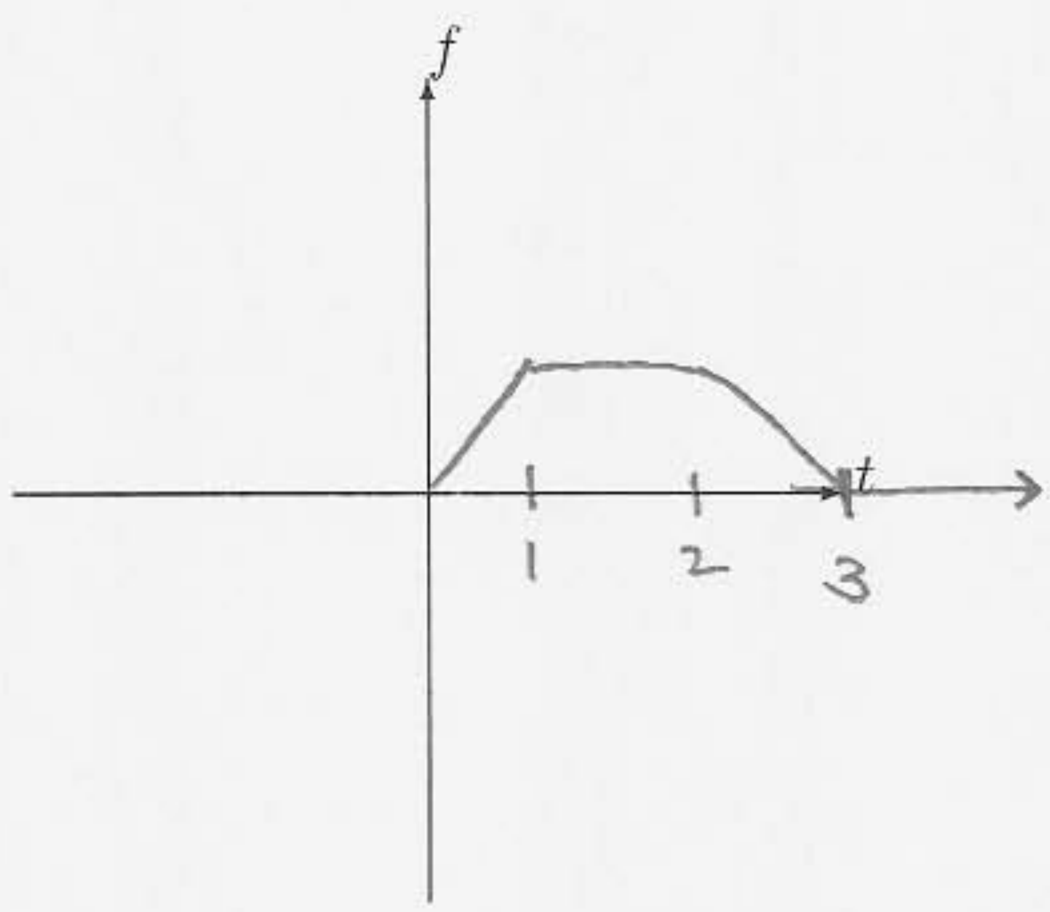
1. (5 Points) Calculate the Laplace transform of $f(t) = te^{-2t}$ using the integral definition.

$$\int_0^\infty te^{-2t}e^{-st}\,dt = \int_0^\infty te^{-(2+s)t}\,dt =$$

$$= -\frac{t}{2+s}e^{-(2+s)t}\Big|_0^\infty - \frac{1}{(2+s)^2}e^{-(2+s)t}\Big|_0^\infty = \frac{1}{(s+2)^2}$$

| $u$ | $dv$ |
|---|---|
| $t$ | $e^{-(2+s)t}$ |
| $1$ | $-\frac{1}{2+s}e^{-(2+s)t}$ |
| $0$ | $+\frac{1}{(2+s)^2}e^{-(2+s)t}$ |

2. (5 Points) Given the following graph of $f$:




Determine an expression for $f(t)$ using step-functions.

$$f(t) = [u_0(t) - u_1(t)]t + 1[u_1(t) - u_2(t)] + -(t-3)[u_2(t) - u_3(t)]$$

3. (10 Points) Given $f(t)$ find $F(s)$:

(a) $f(t) = te^{-2t} + 2t^2 + 4$ $\Rightarrow F(s) = \frac{1}{(s+2)^2} + \frac{4}{s^3} + \frac{4}{s}$

(b) $f(t) = u_4(t)e^{3t}$ $\Rightarrow F(s) = e^{-4s} \mathcal{L}\{e^{3(t+4)}\} = e^{12} e^{-4s} \frac{1}{s-3}$

(c) $f(t) = \frac{1}{2} t \sin(2t)$ $\Rightarrow F(s) = \frac{-2s}{s^2+4}$

$n = 1$

$f_1(t) = \sin(2t) \Rightarrow F_1(s) = \frac{2}{s^2+4} \Rightarrow F_1'(s) = \frac{-4s}{s^2+4}$

4. (10 Points) Given $F(s)$ find $f(t)$:

(a) $F(s) = \frac{1}{(s-1)(s+1)^2} = \frac{A}{s-1} + \frac{B}{(s+1)} + \frac{C}{(s+1)^2} = \frac{A(s+1)^3 + B(s-1)(s+1)^2 + C(s+1)(s-1)}{(s+1)^3(s-1)} =$

$= \frac{A(s+1)^2 + B(s-1)(s+1) + C(s-1)}{(s+1)^2(s-1)}$

$s = 1 \Rightarrow A = \frac{1}{4}$

$s = -1 \Rightarrow C = -1/2$

$s = 0 \Rightarrow 1 = A - B - C \Rightarrow B = A - C - 1$

$= 1/4 + 1/2 - 1$

$= -\frac{1}{4}$

$F(s) = \frac{1}{4}\frac{1}{s-1} - \frac{1}{4}\frac{1}{s+1} - \frac{1}{2}\frac{1}{(s-1)^2} \Rightarrow f(t) = \frac{1}{4}e^{t} + -\frac{1}{4}e^{-t} - \frac{1}{2}te^{t}$

(b) $F(s) = \frac{3s-8}{s^2-4s+13}$

$\frac{3(s-8/3)}{(s-2)^2+9} = \frac{3}{1}\left[\frac{s-2}{(s-2)^2+9} - \frac{2}{9}\frac{3}{(s-2)^2+9}\right]$

$\Rightarrow f(t) = 3e^{2t}\cos(3t) - \frac{2}{3}e^{2t}\sin(3t)$

(c) $F(s) = \frac{5s-2}{s^2+4} = 5\left(\frac{s-2/5}{s^2+4}\right) = 5\left[\frac{s}{s^2+4} - \frac{1}{5}\frac{2}{s^2+4}\right]$

$\Rightarrow f(t) = 5\cos(2t) - \frac{5}{5}\sin(2t)$

5. (20 Points) Solve the following IVP:

$$2y'' + 8y = u_5(t) + \delta_2(t) - 4\cos(2t), \quad y(0) = -2,\ y'(0) = 2 \tag{1}$$

$$\mathcal{L}\{2y''+8y\} = 2sY(s) - 2sy(0) - 2y'(0) + 2Y(s)$$

$$= \frac{e^{-5s}}{s} + e^{-2s} - \frac{4s}{s^2+4}$$

$$Y(s) = \frac{4-4s}{2s^2+8} + \frac{e^{-2s}}{2s^2+8} + e^{-5s}\frac{1}{s}\cdot\frac{1}{2s^2+8} - \frac{2s}{(s^2+4)^2} =$$

$$= \frac{-2s}{s^2+4} + \frac{2}{s^2+4} + \frac{1}{2}e^{-2s}\frac{1}{s^2+4} + \frac{1}{2}e^{-5s}\left(\frac{1}{s}\cdot\frac{1}{s^2+4}\right) - \frac{2s}{(s^2+4)^2}$$

Noting $\frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4} = \frac{As^2+4A+Bs^2+Cs}{s(s^2+4)}$

$\Rightarrow A=1 \quad B=-1 \quad C=0$

Thus,

$$y(t) = -2\cos(2t) + \sin(2t) + \frac{1}{2}u_2(t)\sin(2(t-2)) + \frac{t}{2}\sin(2t) +$$

$$+ \frac{1}{2}u_5(t) - \frac{1}{2}u_5(t)\cos(2(t-5))$$

| Function $f(t)$ | Laplace transform $F(s)$ | Function $f(t)$ | Laplace transform $F(s)$ |
|---|---|---|---|
| $f'(t)$ | $sF(s) - f(0)$ | $e^{at}$ | $\frac{1}{s-a}$ |
| $f''(t)$ | $s^2F(s) - sf(0) - f'(0)$ | $\cos kt$ | $\frac{s}{s^2+k^2}$ |
| $\int_0^t f(\tau)\,d\tau$ | $\frac{F(s)}{s}$ | $\sin kt$ | $\frac{k}{s^2+k^2}$ |
| $e^{at}f(t)$ | $F(s-a)$ | $t^n f(t)$ | $(-1)^n \frac{d^nF}{ds^n}$ |
| $u_c(t)\,f(t-c)$ | $e^{-cs}F(s)$ | $t^n e^{at}$ | $\frac{n!}{(s-a)^{n+1}}$ |
| $\int_0^t f(\tau)g(t-\tau)\,d\tau$ | $F(s)G(s)$ | $e^{at}\cos kt$ | $\frac{s-a}{(s-a)^2+k^2}$ |
| $1$ | $\frac{1}{s}$ | $e^{at}\sin kt$ | $\frac{k}{(s-a)^2+k^2}$ |
| $t$ | $\frac{1}{s^2}$ | $u_c(t)$ | $\frac{e^{-cs}}{s}$ |
| $t^n$ | $\frac{n!}{s^{n+1}}$ | $\delta(t-t_0)$ | $e^{-st_0}$ |