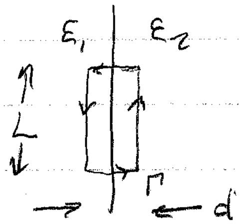
Fresnel Equations - reflection + refraction

- Applies to a step-change in refr. index n
- method: take solutions of wave equation in both media
  use boundary conditions to match $\vec{E}, \vec{H}$
  express reflected, transmitted amplitudes in terms of incident ampl:
- result: ratio $r_\perp, t_\perp$ and $r_\parallel, t_\parallel$
  r, t depend on polarization direction relative to interface.
  r, t are complex, $R \propto |r|^2$, $T \propto |t|^2$ for power or intensity
- Applications
  method is used in calculating $\vec{E}(\vec{r})$ in waveguides, resonators, multilayer structures.
  Fresnel eqns are used to design polarization optics, multilayer coatings, interferometers

Boundary conditions : dielectric interface   re HM 1.8

$\vec{E}$ tangential component continuous   shorthand: $\vec{E}\times\hat{n}$



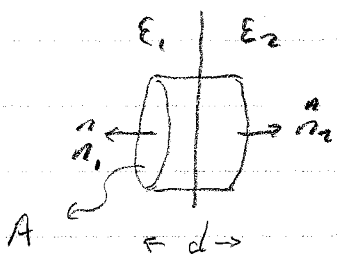
$$\nabla\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$

$$\oint \vec{E}\cdot d\vec{l} = -\frac{1}{c}\frac{\partial}{\partial t}\oint_S \vec{B}\cdot\hat{n}\, da$$

limit $d\to 0$

$$(\vec{E}_1\times\hat{n}_1 - E_2\times\hat{n}_2)L = 0, \quad \vec{E}_1^{\parallel} = \vec{E}_2^{\parallel}$$

$\vec{D}$ normal component continuous.



$$\nabla\cdot\vec{D} = 0 \quad \text{(no free charges)}$$

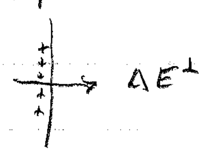
$$\oint_S \hat{D}\cdot\hat{n}\, da = 0 \qquad \text{limit } d\to 0$$

$$= (\vec{D}_1\cdot\hat{n}_1 + \hat{D}_2\cdot\hat{n}_2)A = 0$$

let $\hat{n} = \hat{n}_1 = -\hat{n}_2$

$$\vec{D}_1\cdot\hat{n} = \vec{D}_2\cdot\hat{n} \qquad \text{or } \varepsilon_1 E_1^{\perp} = \varepsilon_2 E_2^{\perp}$$

here, jump in $E^{\perp}$ b/c of bound charges.



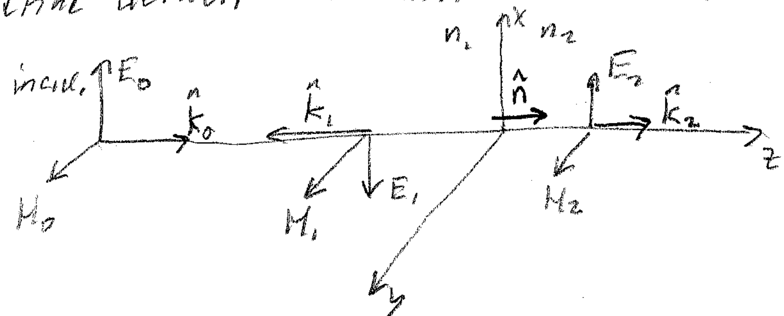
similarly,

$\vec{B}$ normal comp. contin.   (from $\nabla\cdot\vec{B} = 0$)

$\vec{H}$ tang. comp. contin. (no surface currents)

Reflection + transmission: Normal incidence

1) define default orientation of the fields



tangential E, H are continuous at interface

<u>note</u> conventions vary here: sign of r is relative to how fields are set up.

2) write wave solutions in different regions

incident, $\vec{E}_0 = E_0^0 \hat{x}\, e^{i(k_0 n_1 z - \omega t)}$

refl. $\vec{E}_1 = E_1^0 (-\hat{x})\, e^{i(-k_0 n_1 z - \omega t)}$

transm $\vec{E}_2 = E_2^0 \hat{x}\, e^{i(k_0 n_2 z - \omega t)}$

similar for $\vec{H}$ (all in $+\hat{y}$ direction)

set $t=0$: in this formulation, all is time-independent

3) match at boundary $z=0$

$$E_0^0 - E_1^0 = E_2^0$$

$$H_0^0 + H_1^0 = H_2^0$$

take $E \to H$ $\quad \nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \rightarrow \vec{k} \times \vec{E} = \frac{\omega}{c}\mu \vec{H}$

$$k = k_0 n_i = \frac{\omega}{c} n_i \rightarrow H^0 = \frac{n_i}{\mu} E^0$$

let $\mu = 1$

$$\rightarrow n_1 (E_0^0 + E_1^0) = n_2 E_2^0$$

4) solve for E_1^0, E_2^0 in terms of E_0^0

$$n_1 E_0^0 + n_1 E_1^0 = n_2 E_0^0 - n_2 E_1^0$$

reflected $\quad E_1^0 = E_0^0 \dfrac{(n_2 - n_1)}{n_1 + n_2} \equiv r E_0^0$

↳ ampl. reflection coeff.

transm. $\quad E_2^0 = E_0^0 - E_0^0 \dfrac{n_2 - n_1}{n_2 + n_1} = \dfrac{2 n_1}{n_1 + n_2} E_0^0 \equiv t E_0^0$

Comments:

- for $n_2 > n_1$ e.g. air to glass $- r > 0$ but E changes sign on reflection
- for $n_2 < n_1$ $r < 0$ $\therefore$ no sign change in E

At normal incidence, r, t are real

Power / Intensity reflection coefficients

$$R \equiv \frac{\langle \vec{S}_1 \rangle \cdot (-\hat{n})}{\langle \vec{S}_0 \rangle \cdot \hat{n}} = \frac{\frac{c}{8\pi} \mathrm{Re}(\vec{E}_1 \times \vec{H}_1^*) \cdot (-\hat{n})}{\frac{c}{8\pi} \mathrm{Re}(\vec{E}_0 \times \vec{H}_0^*) \cdot \hat{n}}$$ with $H_i^0 = n_i E_i^0$

$$= \frac{n_1 |E_1|^2}{n_1 |E_0|^2} = |r|^2 = \left(\frac{n_2 - n_1}{n_2 + n_1}\right)^2$$

$$T = \frac{n_2}{n_1} \frac{|E_2|^2}{|E_1|^2} = \frac{n_2}{n_1} |t|^2 = \frac{4 n_1 n_2}{(n_1 + n_2)^2}$$

$$R + T = 1 \qquad (\underline{\text{note}}\ r^2 + t^2 \neq 1\ !)$$

the extra factor of $\frac{n_2}{n_1}$ comes from the fact that the intensity is higher in a medium with $n > 1$

$$I = v_{ph} \langle U \rangle = \frac{c}{n} \cdot \varepsilon \frac{|E|^2}{8\pi} = \frac{1}{8\pi} n c |E|^2$$

$U \uparrow$ by n^2

$v_p \downarrow$ by $1/n$