MATH348: INTRODUCTION TO FOURIER TRANSFORMS

You pick the place and I'll choose the time and I'll climb The hill in my own way.

1. Introduction

If f has a Fourier series representation,

$$f(x) = \sum_{n=-\infty}^{\infty} c(\omega_n)e^{i\omega_n x}, \quad \omega_n = n\pi/L, \tag{1}$$

then f is such that $||f||^2 = \langle f, f\rangle < \infty$ where $\langle f, g\rangle = \int_a^b f(x)f^*(x)dx$. The orthogonality of

$$\mathcal{B} = \left\{\ldots, e^{-i\omega_3 x}, e^{-i\omega_2 x}, e^{-i\omega_1 x}, 1, e^{i\omega_1 x}, e^{i\omega_2 x}, e^{i\omega_3 x}, \ldots\right\} \tag{2}$$

can be used to show that

$$\frac{1}{2L}\int_a^b |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c(\omega_n)|^2 \propto E_{\text{total}}, \tag{3}$$

which relates the Fourier coefficients

$$c_n = c(\omega_n) = \frac{1}{2L}\left\langle f, e^{i\omega_n x}\right\rangle, \tag{4}$$

to the total single-cycle energy in f. Mathematically we might say that the set of functions

$$L^2(a,b) = \left\{f : (a,b) \in C^0(a,b) : ||f||^2 = \int_a^b |f(x)|^2 dx < \infty\right\}, \tag{5}$$

where $C^0(a,b)$ is the set of piecewise continuous functions defined on (a,b), forms a vector space under addition of functions and multiplication of functions by scalars. Notice that this implies that points in this space must have finite collective oscillator energy; this is what is meant by f being *reasonable.* Now, there are two striking deficiencies in the Fourier series:

1. The function f, represented by a Fourier series, must be periodic or made to be periodic by repetition of the data on the principle domain, (a,b), into the rest of the real-line, $(-\infty,\infty)$. That is, whatever a Fourier series represents is naturally periodic.
2. This next issue is intimately tied to the previous and that is, all modes in a Fourier series are related through their frequencies. Specifically, modes of a Fourier series have frequencies that are integer multiples of a base frequency. This means that though higher-frequency modes oscillate more frequently, all modes will still repeat themselves on the same width as the fundamental mode.[1] While this makes that series periodic, the deficiency I would like to point out is

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[1]The fundamental mode is that mode with lowest non-zero frequency.

that there are frequencies between the gaps $\Delta\omega = \omega_{n+1} - \omega_n$ that are not being used, i.e. have no energy.

It turns out that by forcing your way into all of the nooks and crannies within the frequency–gaps, defined by $\Delta\omega = \omega_{n+1} - \omega n$, both limitations will be resolved. This extension/generalization is called the Fourier transform and defines a basis for a space of functions f may or may not be periodic.[2]

2. The Fourier Transform

To make use of those frequencies a Fourier series does not, we consider the limit $L \to \infty$. This limit implies that $\Delta\omega \to 0$ and that there are no frequency gaps. So, in this limit we should be allowed to use all frequencies $\omega \in (-\infty, \infty)$. All we must do is take the limit of a Fourier series as the width of the principle domain becomes infinite. Before this limit we prepare the Fourier series,

$$f_L(x) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{2L} \int_a^b f_L(v) e^{-i\omega_n v} dv \right] e^{i\omega_n x} \tag{6}$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \underbrace{\left[\frac{1}{\sqrt{2\pi}} \int_a^b f_L(v) e^{-i\omega_n v} dv \right] e^{i\omega_n x}}_{=F(\omega_n),\text{Function of } \omega_n} \Delta\omega \tag{7}$$

First, it is a typical mathematical convention to split the 2π, from the $2L$, across the summand and the summation.[3] What is more important to notice is that the summand is a function of ω_n. With this in mind, the limit $L \to \infty$ defines a Riemann sum[4] in the ω–variable. So, as $L \to \infty$ we have that $\Delta\omega \to 0$, $\omega_n \to \omega$ and $\displaystyle\sum_{n=-\infty}^{\infty} F(\omega_n)\Delta\omega \to \int_{-\infty}^{\infty} F(\omega) d\omega$. The only thing that remains is dealing with the emerging improper integral,

$$\lim_{L\to\infty} \int_a^b f_L(v) e^{-i\omega_n v} dv, \quad b - a = 2L. \tag{9}$$

As $L \to \infty$ we integrate over all of $\mathbb{R}$, which increases the risk of divergence. To avoid divergence we demand that f_L be absolutely integrable on $\mathbb{R}$. That is,

$$\lim_{L\to\infty} \int_{-\infty}^{\infty} |f_L(x)| dx < \infty, \tag{10}$$

which implies that the total "area under the curve" that f_L defines is finite, even in the limit. Since $e^{i\omega_n v}$ will only scale f_L, the risky integral is finite under the limit,

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[2]To be quite technical, the space isn't L^2 because of an assumption that must be made in the derivation of Fourier transform. Keep an eye out!

[3]This is not always the case. Different texts will do this part differently. In other courses you should consult your text to make sure you are using the same conventions.

[4]Recall from Calculus I the concept,

$$\lim_{N\to\infty} \sum_{n=1}^{N} f(x_n)\Delta x = \int_a^b f(x) dx, \tag{8}$$

where $x \in (a, b)$ and $\Delta x = x_{n+1} - x_n = (b - a)/N$.

$L \to \infty$. Finally, we have that

$$f(x) = \lim_{L\to\infty} f_L(x) \tag{11}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v) e^{-i\omega v} dv \right] e^{i\omega x} dx, \tag{12}$$

which we notice defines the Fourier transform pair,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega, \tag{13}$$

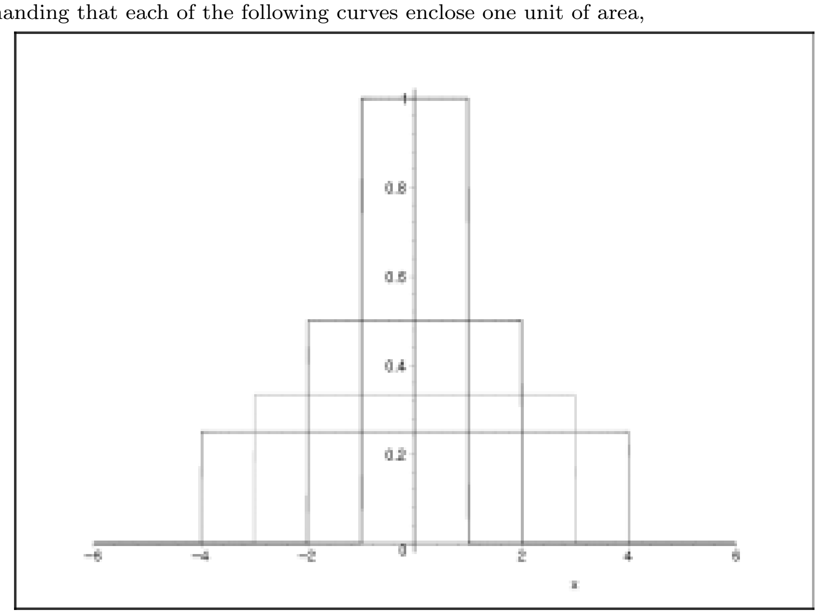
$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \tag{14}$$

The first of the two formulae, often called a Fourier integral representation, should be thought of as Eq. (1) where we have allowed the use of all available frequencies. While the function, f, need not be periodic, it can still be thought of as the interference of primitive waves $e^{i\omega x}$ of different angular–frequency. The second formula, called the Fourier transform of f, can be thought of as a Fourier coefficient, Eq. (4), where we have compared f to $e^{-i\omega x}$ in order to know how much amplitude, $\hat{f}$, is needed for each frequency, ω. With this analogy we find a similar expression for the total "energy" of the signal f, $\int_{-\infty}^{\infty} |\hat{f}|^2 d\omega \propto E$.

From the Fourier transform, it is possible to get another perspective on Fourier series. First, we have to accept the concept of a Dirac delta "function,"[5] whose properties are:

1. Ideal Localization: $\delta(x - x_0) = 0$ for all $x \neq x_0 \in \mathbb{R}$
2. Unit Area: $\int_{-\infty}^{\infty} \delta(x - x_0) dx = 1$

[5] It must be understood that what we write is not a function, but a distribution or generalized function. Language aside, there is no function that will have the listed behavior, which is better thought of as a sequence of integrals that limits to the desired formulae. Remember the concept of demanding that each of the following curves enclose one unit of area,



but the width of the rectangles tends to zero?

3. Ideal Measurement: $\int_{-\infty}^{\infty} \delta(x-x_0)f(x)dx = f(x_0)$

From this we see that $\mathcal{F}\{\delta(t)\} = \frac{1}{\sqrt{2\pi}}$ and conclude that it would take an infinite amount of energy to ideally localize, in time, a transmitted signal. Regardless, it makes sense to pursue this idea further. Consider now the following transform,

$$\mathcal{F}\{\cos(\omega_0 t)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\cos(\omega_0 t)e^{-i\omega t}dt \tag{15}$$

$$\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\cos(\omega_0 t)\cos(\omega t)dt, \quad \omega_0 \in \mathbb{R}. \tag{16}$$

It should be clear that whatever comes out of this will be even in ω. However, due to the bound of integration, it is unclear what comes out. We do know two things:

1. $\hat{f}$ should be even, just as the input function is.
2. The input function, $\cos(\omega_0 t)$, has two frequencies of importance, $\pm\omega_0$.

Based on this we can guess the following form for $\hat{f}$,

$$\hat{f}(\omega) = \sqrt{\frac{\pi}{2}}\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right] \tag{17}$$

and upon transformation we find that $\mathcal{F}^{-1}\left\{\hat{f}\right\} = \cos(\omega_0 t)$. So, using the concept of a Dirac delta "function," we can recover a cosine function. We can take this a little further. Consider,

$$\mathcal{F}^{-1}\left\{\sum_{n=-\infty}^{\infty}\sqrt{2\pi}c_n\delta(\omega-\omega_n)\right\} = \int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty}c_n\delta(\omega-\omega_n)e^{i\omega x}d\omega \tag{18}$$

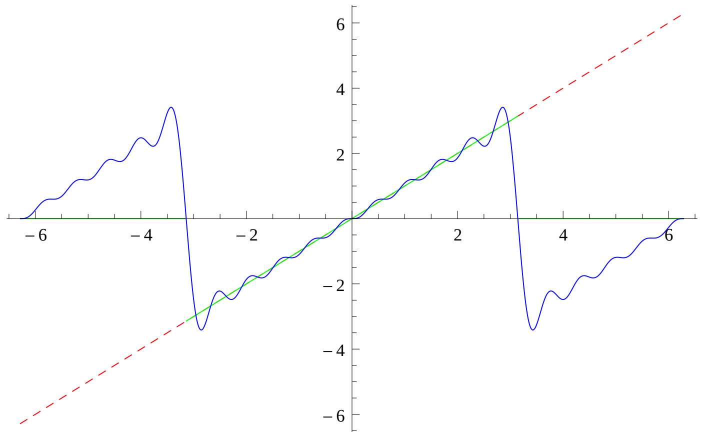
$$= \sum_{n=-\infty}^{\infty}c_n e^{i\omega_n x}, \tag{19}$$

where $\omega_n = n\pi/L$. This is nothing more than a complex Fourier series. So, we see that if one ideally localizes energy to specific frequencies that are evenly–spaced by widths $\Delta\omega = \pi/L$, then it is possible to recover a Fourier series from a Fourier integral. So from this we learn, the Fourier integral/transform is a more general object than Fourier series and the concept of a periodic function is pretty ideal.

To conclude, let's consider a more realistic signal defined by,

$$f(t) = \begin{cases} t, & t \in (-\pi,\pi) \\ 0, & t \notin (-\pi,\pi) \end{cases} \tag{20}$$

The following graph shows this function (green), a ten-mode Fourier series approximation to the repetition of this graph into the whole space (blue) and the function $y = x$ (red–dashed).



So, if we care only about the data, f, on the 2π–width, $(-\pi, \pi)$, then we have two ways to represent it:

1. Fourier series representation: Here the data on $(-\pi, \pi)$ is repeated into the rest of the space and the familiar sawtooth wave is the result.
2. Fourier integral representation: Here the data is made to be zero outside of $(-\pi, \pi)$ and a single cycle of a sawtooth wave is the result.

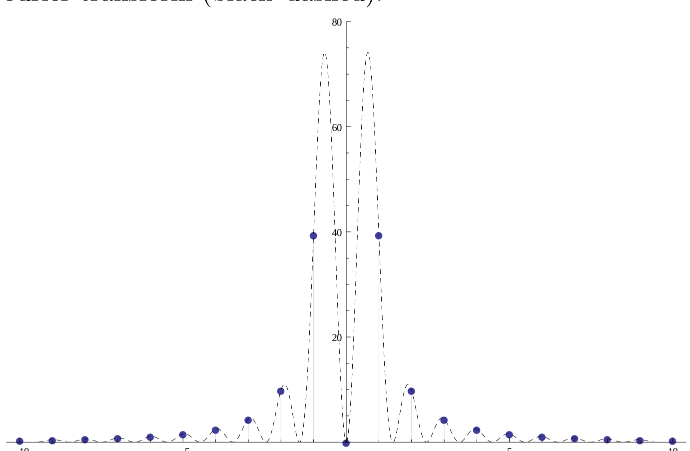
The Fourier coefficients for the sawtooth wave are given by

$$c(\omega_n) = \frac{i(-1)^n}{n} \tag{21}$$

and the Fourier transform for the single cycle is given by

$$\hat{f}(\omega) = i\sqrt{\frac{2}{\pi}} \frac{(-\pi\omega\cos(\pi\omega) + \sin(\pi\omega))}{\omega^2}. \tag{22}$$

The following graph shows the power-spectrum for both the Fourier series (blue–dots) and Fourier transform (black–dashed).



Notice the Fourier coefficients are ideally localized points, which correspond to points on the graph of the Fourier transform. Altogether we conclude that a Fourier series and coefficients, Eq. (1), (4), are really just a special case of the Fourier transform pair, Eq. (13)–(14), where we have ideally localized energy/amplitude to exact frequencies, which are multiples of some common base frequency. The outcome is that the graph of a Fourier series is the periodic extension of the data on $(-\pi, \pi)$. Both techniques enable the user to simultaneously consider the perspectives amplitude as a function of space/time and amplitude/energy as a function of frequency. The duality is important in signal analysis, wave propagation, optics and quantum mechanics.

3. Things to do

1. Show that $\mathcal{F}\left\{\delta(x-x_0)\right\} = \dfrac{e^{-i\omega x_0}}{\sqrt{2\pi}}$ where $x_0 \in \mathbb{R}$.
2. If you haven't shown that $\mathcal{F}\left\{A(U(t+L)-U(t-L))\right\} = A\sqrt{\dfrac{2}{\pi}}\dfrac{\sin(L\omega)}{\omega}$ where U is the Heavyside step-function, $U(t-a) = \begin{cases} 1, & t>0 \\ 0, & t<0 \end{cases}$, then you should.
3. Using the previous problem, show that $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(L\omega)}{\pi\omega} d\omega = 1$
4. Show that the Fourier transform of Eq. (20) is Eq. (22).
5. Show that $\mathcal{F}^{-1}\left\{\sqrt{\dfrac{\pi}{2}}\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]\right\} = \cos(\omega_0 t)$ and that $\mathcal{F}^{-1}\left\{i\sqrt{\dfrac{\pi}{2}}\left[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\right]\right\} = \sin(\omega_0 t)$ where $\omega_0 \in \mathbb{R}$.
6. Show that $\mathcal{F}\left\{e^{-k_0|x|}\right\} = \sqrt{\dfrac{2}{\pi}}\dfrac{k_0}{k_0^2+\omega^2}$ where $k_0 \in \mathbb{R}^+$.
7. Show that if f is even and has a Fourier transform then it's Fourier transform pair becomes,

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}(\omega)\cos(\omega x)d\omega \tag{23}$$

$$\hat{f}(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(\omega x)dx \tag{24}$$

8. Show that if f is odd and has a Fourier transform then it's Fourier transform pair becomes,

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}(\omega)\sin(\omega x)d\omega \tag{25}$$

$$\hat{f}(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(\omega x)dx \tag{26}$$

9. Show that $\mathcal{F}\left\{f(x+c)\right\} = e^{ic\omega}\mathcal{F}\left\{f\right\}$ where $c \in \mathbb{R}$.
10. Show that $\mathcal{F}\left\{f(ax)\right\} = \hat{f}\left(\omega/a\right)/a$ where $a \in \mathbb{R}^+$.
11. Show that $\mathcal{F}\left\{f'\right\} = i\omega\mathcal{F}\left\{f\right\}$.
12. Show that $\mathcal{F}\left\{f*g\right\} = \sqrt{2\pi}\hat{f}(\omega)\hat{g}(\omega)$ where $(f*g)(x) = \displaystyle\int_{-\infty}^{\infty} f(x-p)g(p)dp$.
13. Show that $\mathcal{F}\left\{e^{-ax}U(x)\right\} = \dfrac{1}{\sqrt{2\pi}(a+i\omega)}$ where $a \in \mathbb{R}^+$.
14. Find the steady–state solution to $y'+y=f(t)$ for $0<t<\infty$.

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