Lecture 4: Maxwell eqns in matter, energy + momentum in fields

We now have time-dependent Maxwell equations + potentials in vacuum. The only bits of cleanup remaining are to generalize the Maxwell equations to include matter, and to check up on our old boundary conditions.

As in statics, we can write the Maxwell equations in matter with no modifications as long as we recall that the source terms ρ and $\vec{J}$ refer to <u>all</u> sources, free and bound:

$$\nabla \cdot E(t) = \rho_t(t)/\varepsilon_0$$

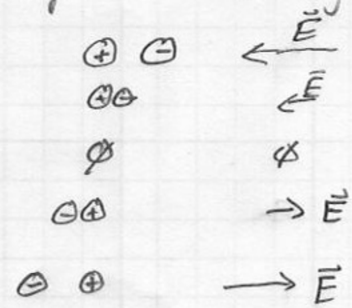
$$\nabla \cdot B(t) = 0$$

$$\nabla \times \vec{E}(t) = -\frac{\partial \vec{B}(t)}{\partial t}$$

$$\nabla \times \vec{B}(t) = \mu_0 \vec{J}_t(t) + \varepsilon_0 \mu_0 \frac{\partial \vec{E}(t)}{\partial t}$$

The fact that $\vec{E}$, $\vec{B}$, ρ, and $\vec{J}$ can be functions of time doesn't really change anything, with one important exception: There's a new kind of $\vec{J}$.

If $\vec{E}$ can change, then $\vec{P}$ can change. And $\vec{P}$ represents a charge separation. So if, for example we have an atom in the presence of an oscillating E-field, we'll get a little something like the following:



That looks a lot like a moving charge, which is a current. And it <u>is</u> a kind of current that we get only in dynamics, the polarization current

$$\vec{J}_p = \frac{\partial \vec{P}}{\partial t}$$

So if we define $\vec{J}_{total} = \vec{J}_{free} + \vec{J}_{bound} + \vec{J}_p$, the Maxwell equations hold as written.

Now, as in statics, we can introduce $\vec{D}$ + $\vec{H}$ to rewrite Gauss's law + the Ampere-Maxwell eqn entirely in terms of free sources. Gauss's law is a freebie since neither the form of it nor ρ_{total} has changed:

$$\nabla \cdot \vec{D}(t) = \rho_f(t)$$

You can probably guess what'll happen to the Ampere-Maxwell eqn. In fact, pause and take a guess.

We'll crank it out in detail to be safe. Start with:

$$\nabla \times \vec{B} = \mu_0 (\vec{J}_f + \vec{J}_b + \vec{J}_p) + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$ Now insert $\vec{J}_b = \nabla \times \vec{M}$, $\vec{J}_p = \frac{\partial \vec{P}}{\partial t}$

$$\Rightarrow \quad \nabla \times \vec{B}/\mu_0 = \cancel{\mu_0} \vec{J}_f + \cancel{\mu_0} \nabla \times \vec{M} + \cancel{\mu_0} \frac{\partial \vec{P}}{\partial t} + \varepsilon_0 \cancel{\mu_0} \frac{\partial \vec{E}}{\partial t}$$

$$\Rightarrow \quad \nabla \times (\vec{B}/\mu_0 - \vec{M}) = J_f + \frac{\partial}{\partial t}(\varepsilon_0 \vec{E} + \vec{P})$$ And $\vec{B}/\mu_0 - \vec{M} \equiv \vec{H}$, $\varepsilon_0 \vec{E} + \vec{P} \equiv \vec{D}$

$$\Rightarrow \quad \boxed{\nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}}$$

Like I said: kind of what we expect. Same law but with $\vec{B} \rightarrow \vec{H}$, $\vec{E} \rightarrow \vec{D}$, $\vec{J}_t \rightarrow \vec{J}_f$, and some constants get absorbed.

Now let's take a look at our boundary conditions, which we've written as:

(1) $E_{2\perp} - E_{1\perp} = \sigma_t / \varepsilon_0$ or $D_{2\perp} - D_{1\perp} = \sigma_f$ (1b)

(2) $B_{2\perp} - B_{1\perp} = 0$

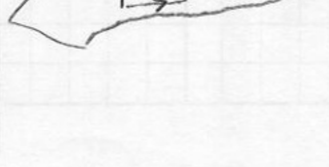
(3) $\vec{E}_{2\parallel} - \vec{E}_{1\parallel} = 0$

(4) $\vec{B}_{2\parallel} - \vec{B}_{1\parallel} = \mu_0 \vec{K} \times \hat{n}$ or $\vec{H}_{2\parallel} - \vec{H}_{1\parallel} = \vec{K}_f \times \hat{n}$ (4b)

(1) and (1b) were derived from flux arguments based on Gauss's Law. We looked at the flux through a vanishingly thin box encasing a surface.

Nothing about this argument was sensitive to the time dependence (or lack thereof) of $\vec{E}$ + σ. So (1), (1b), and also (2) are unchanged.

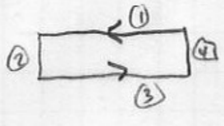
What of (3)? There we did invoke something relevant. We looked at $\oint \vec{E} \cdot d\vec{l}$ along a loop straddling a surface, noted that the curl of $\vec{E}$ is zero, took that to mean $\oint \vec{E} \cdot d\vec{l} = 0$, and deduced that the E-field along the top + bottom portions of the loop must be the same.



But now the curl of $\vec{E}$ isn't zero. Is this bad?

No, actually. That turns out not to matter. We simply proceed as follows:

$$\oint \vec{E} \cdot d\vec{l} = \int E_{1\parallel}\, dl_1 - \int E_{2\parallel}\, dl_2$$ (for infinitesimal sides ② & ④)



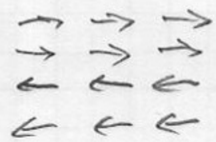
And $\oint \vec{E} \cdot d\vec{l} = -\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A}$

So as long as $\partial \vec{B}/\partial t$ is finite (hint: it is), for an infinitesimally thin loop, $\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A}$ and thus $\oint \vec{E} \cdot d\vec{l}$ is zero, so we still get

$$\boxed{E_{1\parallel} = E_{2\parallel}}$$

Here's the physics: In dynamics, $\vec{E}$-fields can have curl, but not all curly fields can be $\vec{E}$-fields.

Curly fields such as



that have discontinuities are forbidden.

In a similar way, we find that the boundary conditions for $B_\parallel$ and $H_\parallel$ are unchanged from the static case.

The short, short version: Adding time dependence doesn't change the field boundary conditions. And considering the Maxwell equations in matter barely changes anything: We just have to add the polarization current J_p to J_t.

Field energy + momentum, Poynting vector

In the past we've claimed without proof that the energy densities associated with electric and magnetic fields are

$$u_E = \tfrac{1}{2}\epsilon_0 E^2 \quad \text{and} \quad u_B = \frac{1}{2\mu_0} B^2$$

We can now prove this and develop statements about energy transfer. Let's start by recalling the continuity equation:

$\nabla \cdot \vec{J} = -\partial\rho/\partial t$ This is a statement of charge conservation: If the current density $\vec{J}$ is diverging, it must be that the charge density in the region is changing.

Let's write an analogous statement about energy. Let $\vec{S}$ be the energy flux associated with some fields. If the flux is diverging, it must be that the energy density in the region is changing:

$\vec{\nabla} \cdot \vec{S} = \frac{-\partial u}{\partial t} - \frac{\partial u_K}{\partial t}$ There are two energy densities: One representing energy intrinsic to the fields (u), and one representing kinetic energy given to charges by the fields (u_K)

Now, kinetic energy is imparted by way of $K = \int \vec{F} \cdot d\vec{l}$
or $K = \int q\vec{E} \cdot d\vec{l}$ (none for $\vec{B}$)

$$\Rightarrow \quad \frac{dK}{dt} = q\vec{E} \cdot \frac{d\vec{l}}{dt} = q\vec{E} \cdot \vec{v} = \vec{E} \cdot (q\vec{v})$$

We can write this in terms of densities by dividing each side by volume and getting (since $\vec{J} = \rho\vec{v}$):

$$\frac{\partial u_K}{\partial t} = \vec{E} \cdot \vec{J} \qquad \text{And thus} \qquad \vec{\nabla} \cdot \vec{S} = -\frac{\partial u}{\partial t} - \vec{E} \cdot \vec{J}$$

Time to do some massaging. We'd like to get this all in terms of fields, so from Ampere-Maxwell we get $\vec{J} = \frac{1}{\mu_0}(\nabla \times \vec{B}) - \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

$$\Rightarrow \nabla \cdot \vec{S} = -\frac{\partial u}{\partial t} - \vec{E} \cdot \left(\frac{1}{\mu_0} \nabla \times \vec{B}\right) + \epsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t}$$

We have to be a bit careful with the third term. E may be a function of t, so we can't just slide $\partial/\partial t$ around. I claim

$\vec{E} \cdot \frac{\partial \vec{E}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t} E^2$ because $\vec{E} \cdot \partial\vec{E}/\partial t = E_x \, \partial E_x/\partial t + \ldots$

$$\text{and} \quad \frac{1}{2}\frac{\partial}{\partial t}(E(t))^2 = \frac{1}{2}\frac{\partial}{\partial t}\left(E_x^2 + E_y^2 + E_z^2\right)$$

$$= \frac{1}{2} \cdot 2E_i \frac{\partial E_i}{\partial t} \quad \text{(implied summation)}$$

Also, let's hit the second term with a vector identity:

$$\vec{E}\cdot(\vec{\nabla}\times\vec{B}) = -\nabla\cdot(\vec{E}\times\vec{B}) + \vec{B}\cdot(\nabla\times\vec{E})$$
$$= -\nabla\cdot(\vec{E}\times\vec{B}) + \vec{B}\cdot\left(-\frac{\partial\vec{B}}{\partial t}\right)$$

Mashing it all together,

$$\nabla\cdot\bar{S} = -\frac{\partial u}{\partial t} + \nabla\cdot\left(\frac{\vec{E}\times\vec{B}}{\mu_0}\right) - \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0}B^2\right)$$

Or

$$\nabla\cdot\bar{S} + \frac{\partial u}{\partial t} = \nabla\cdot\left(\frac{\vec{E}\times\vec{B}}{\mu_0}\right) - \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0}B^2\right)$$

Which is satisfied in general when $\boxed{\bar{S} = \frac{\vec{E}\times\vec{B}}{\mu_0} \quad \text{and} \quad u_{field} = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0}B^2}$

Poynting's theorem is $\nabla\cdot\bar{S} = -\frac{\partial u}{\partial t} - \vec{E}\cdot\vec{J}$ with the above for $\bar{S}$ and u.

$\bar{S}$ is called the Poynting vector, which is a vector describing an energy flux, or an energy transfer. $\bar{S}$ has units of W/m^2, which you may recall as an intensity.

Finally, we shall simply state two results regarding the momentum and angular momentum associated with E + B fields. These relationships will fall out naturally in relativity, where we discover that momentum and energy have the same relationship as space and time (whoa).

We have for linear momentum density: $p = \mu_0\varepsilon_0\bar{S} = \varepsilon_0(\vec{E}\times\vec{B})$

and angular mom. density: $l = \vec{r}\times p = \varepsilon_0(\vec{r}\times\vec{E}\times\vec{B})$

So neither an $\vec{E}$ nor a $\vec{B}$ by itself has momentum, but E's + B's together at right angles do. Kind of an interesting constraint.