MATH-300 - Foundations of Adv. Mathematics **Field 2009**
Midterm 1 **Due: Thursday May 28, 5:00pm**

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1. In class, we proved the Triangle Inequality using an observation regarding absolute value. This time, however, we are going to take the long way to the proof. Using cases, prove

$$\text{For all real numbers } x \text{ and } y,\ |x+y| \le |x| + |y|$$

Proof. To prove the Triangle Inequality we will consider the following cases:

Case 1: $x \ge 0$, $y \ge 0$.
$x \ge 0,\ y \ge 0 \Rightarrow x + y \ge 0$
Thus, $|x+y| = x + y = |x| + |y|$.
Therefore, $|x+y| \le |x| + |y|$.

Case 2: $x \le 0$, $y \le 0$.
$x \le 0,\ y \le 0 \Rightarrow x + y \le 0$
Thus, $|x+y| = -(x+y) = -x + -y = |x| + |y|$.
Therefore, $|x+y| \le |x| + |y|$.

Case 3: $x < 0$, $y \ge 0$ and $|x| > |y|$.
Note: Without loss of generality, in the cases where x and y differ in sign, we can assume $x < 0$.
Thus, $x < 0$ and $|x| > |y| \Rightarrow |x+y| = -(x+y) = -x - y$. Similarly, $|x| + |y| = -x + y$. Since $y \ge 0$, we have

$$-y < y \Rightarrow -x - y < -x + y.$$

Therefore, $|x+y| \le |x| + |y|$.

Case 4: $x < 0$, $y \ge 0$ and $|x| < |y|$.
Thus, $x < 0$ and $|x| < |y| \Rightarrow |x+y| = x+y$. Similarly, $|x|+|y| = -x + y$. Since $x < 0$, we have

$$x < -x \Rightarrow x + y < -x + y.$$

Therefore, $|x+y| \le |x| + |y|$.

Therefore, in general, for all real numbers, x and y,

$$|x+y| \le |x| + |y|$$

□

2. Define A as the average of the n numbers, $x_1, x_2, \ldots, x_n$. Prove that at least one of the $x_1, \ldots, x_n$ is greater than or equal to A.

Proof. Proof by contradiction.
Assume that it is not the case that at least one of the $x_1, \ldots, x_n$ is greater than or equal to A. In other words, $\forall i \; 1 \leq i \leq n, \; x_i < A$.
By definition of the arithmetic mean

$$A = \frac{x_1 + \ldots + x_n}{n}$$

Since $x_i < A$ for all $a \leq i \leq n$, we know

$$A = \frac{x_1 + \ldots + x_n}{n} < \frac{A + \ldots + A}{n} = \frac{nA}{n} = A$$

Thus we have $A < A$, an impossibility.
Therefore, by contradiction, it must be the case that at least one x_i is greater than or equal to the arithmetic mean, A. □

3. Let a and b be integers with $a \neq 0$. If a does not divide b, then the equation $ax^3 + bx + (b + a) = 0$ does not have a solution that is a natural number.
(Hint: It may be necessary to factor a sum of cubes. Recall that $u^3 + v^3 = (u + v)(u^2 - uv + v^2)$.)

Proof. Proof by contrapositive.
If the equation $ax^3 + bx + (b + a) = 0$ has a natural number solution then $a|b$.
Let $n \in \mathbb{N}$ be a solution to the given equation

$$\Rightarrow an^3 + bn + (b + a) = 0$$

Consider,

$$\begin{aligned}
an^3 + bn + (b + a) &= a\left(n^3 + \frac{b}{a}n + (1 + \frac{b}{a})\right) \\
&= a\left((n^3 + 1) + (\frac{b}{a}n + \frac{b}{a})\right) \\
&= a\left((n^3 + 1) + \frac{b}{a}(n + 1)\right) \\
&= a\left((n + 1)(n^2 - n + 1) + \frac{b}{a}(n + 1)\right) \\
&= a\left((n + 1)(n^2 - n + (1 + \frac{b}{a}))\right)
\end{aligned}$$

Since $n \in \mathbb{N}$, $n+1 \neq 0 \Rightarrow (n^2 - n + (1 + \frac{b}{a})) = 0$.
Also, since $n \in \mathbb{N}$, $n^2 - n + 1 \in \mathbb{N}$. Thus, $\frac{b}{a} = -(n^2 - n + 1)$ is also a natural number $\Rightarrow a|b$.
Therefore, if a does not divide b, then the equation $ax^3+bx+(b+a) = 0$ does not have a solution that is a natural number. □

4. Prove the following proposition
For all sets A, B, and C that are subsets of some universal set, if

$$A \cap B = A \cap C \text{ and } A^c \cap B = A^c \cap C, \text{ then } B = C.$$

Proof.

$B \subseteq C$

Let $x \in B$. Then with respect to set A, we have 2 cases:

(a) $x \in A$
$x \in A \Rightarrow x \in A \cap B = A \cap C \Rightarrow x \in C$.
Thus, $B \subseteq C$.

(b) $x \notin A$
$x \notin A \Rightarrow x \in A^c \Rightarrow x \in A^c \cap B = A^c \cap C \Rightarrow x \in C$. Thus $B \subseteq C$.

Therefore, $B \subseteq C$.

$C \subseteq B$

Similarly to that given above, let $x \in C$. With respect to set A, either $x \in A$ or $x \in A^c$. In either case, we can conclude that $x \in B$

Thus, since $B \subseteq C$ and $C \subseteq B$, $B = C$. □