Resonators and stability

Examples of resonators

Unfolded resonator and ABCD description

Stability

- Ray picture
- Gaussian beam picture

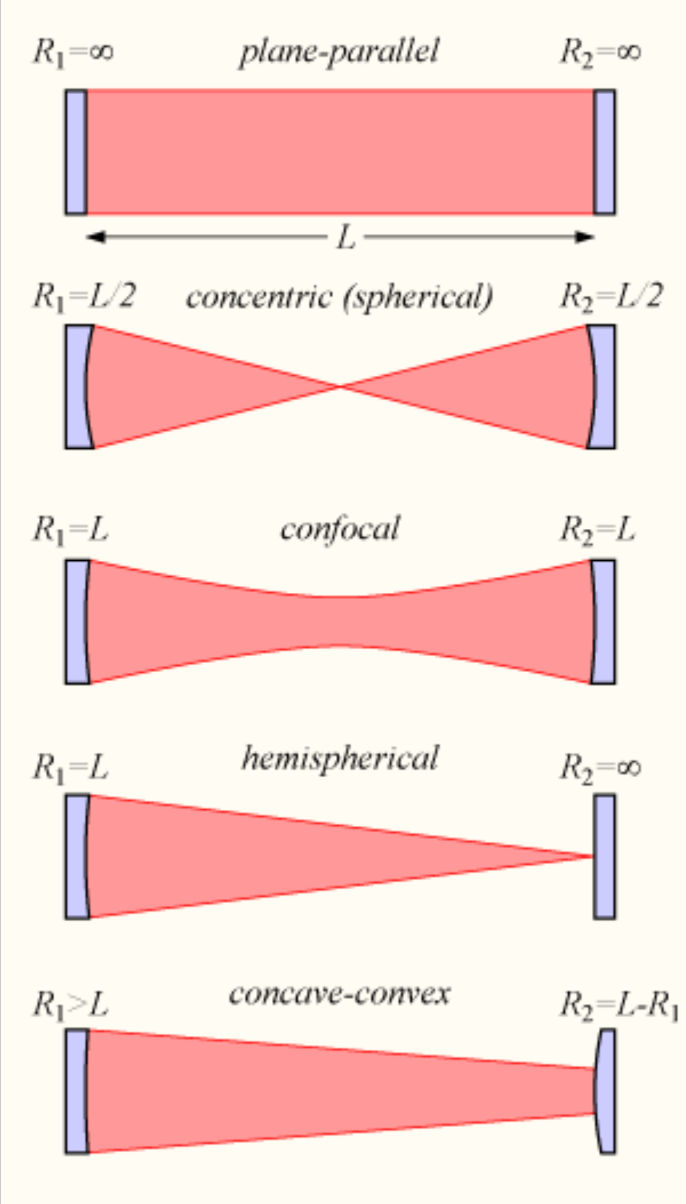
2 mirror resonators and the stability map

Analysis of resonators

- beam sizes

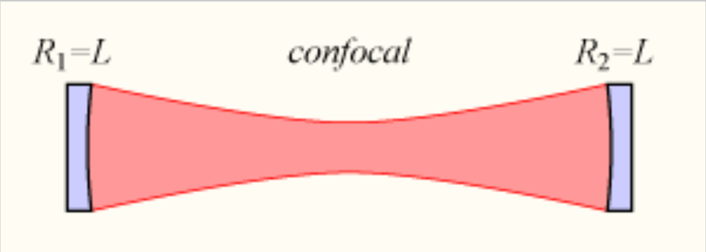
Resonators

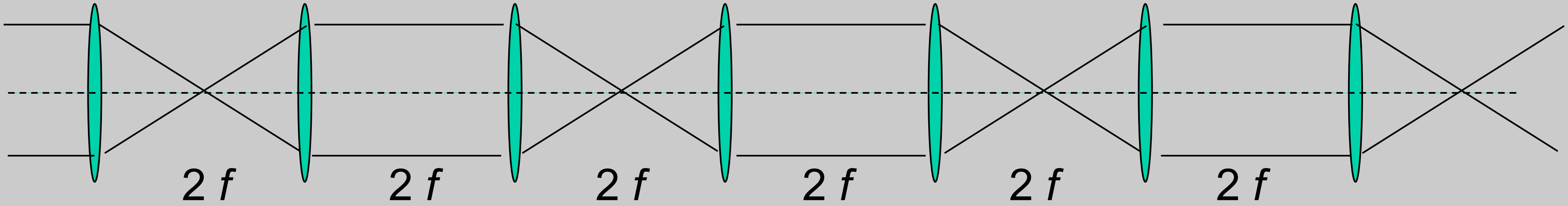
- Resonators provide feedback for the photons to build up by passing through the gain medium
- Curved mirrors are typically used to control the beam size inside the gain medium
- Types of resonators
  - Many resonators have more than two mirrors, but most can be mapped onto a two-mirror system.



Periodic lens model

- A resonator can be "unfolded" by modeling the curved mirrors as ideal lenses

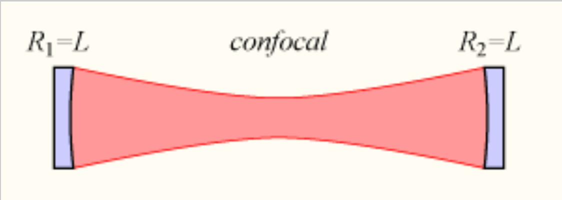




- Are there rays that will stay confined?
- If so, resonator is ***stable***.

Resonator ABCD model

- Build a ABCD matrix model of the periodic lens sequence
  - First get a *round-trip* matrix



$$M_{RT}(f,L)=\begin{pmatrix} A & B \\ C & D \end{pmatrix}=M_L(f)\cdot M_T(L)\cdot M_L(f)\cdot M_T(L)$$

$$=\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

  - Free to choose starting point
  - Focal length ***f*** and mirror separation ***L*** can vary

$$\begin{pmatrix} r_2 \\ r_2' \end{pmatrix}=\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} r_0 \\ r_0' \end{pmatrix}$$

After 2 round trips

Resonator stability: ray picture

- Will a ray stay trapped?
- Look at whether r_n and r'_n stay finite as n goes to infinity

$$\begin{pmatrix} r_n \\ r'_n \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^n \begin{pmatrix} r_0 \\ r'_0 \end{pmatrix}$$

- Method:
  - Diagonalize matrix: $M_{RT} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = U \begin{pmatrix} \lambda_a & 0 \\ 0 & \lambda_b \end{pmatrix} U^{-1}$
  - then $M_{RT}{}^n = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^n = U \begin{pmatrix} \lambda_a & 0 \\ 0 & \lambda_b \end{pmatrix} U^{-1} U \begin{pmatrix} \lambda_a & 0 \\ 0 & \lambda_b \end{pmatrix} U^{-1} \cdots$

$$= U \begin{pmatrix} \lambda_a & 0 \\ 0 & \lambda_b \end{pmatrix}^n U^{-1} = U \begin{pmatrix} \lambda_a{}^n & 0 \\ 0 & \lambda_b{}^n \end{pmatrix} U^{-1}$$

Stability condition

- The ray will stay trapped (stable resonator) if

$$|\lambda_a| \le 1 \quad |\lambda_b| \le 1 \qquad\qquad \frac{1}{|\lambda_a|} \le 1 \quad \frac{1}{|\lambda_b|} \le 1$$

Reverse propagation

- Therefore matrix eigenvalues must satisfy $|\lambda_a| = |\lambda_b| = 1$
- Property of ABCD: $\det M_{RT} = \lambda_a \lambda_b = 1$

$$\therefore \lambda_a = e^{i\theta}, \lambda_b = \lambda_a{}^*$$

- Trace of M is invariant upon rotation of matrix:

$$\mathrm{Tr}\, M_{RT} = \lambda_a + \lambda_b = A + D$$

$$= e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

- Finally stability condition is: $\boxed{-1 \le \frac{A+D}{2} \le 1}$

Some properties of ABCD matrices

1. Determinant = 1 if start and end points are in the same medium (same refr. Index)
   - Example: $\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$
   - Counter example: dielectric interface $\begin{pmatrix} 1 & 0 \\ 0 & n_1/n_2 \end{pmatrix}$
   - Therefore, det M = 1, but note that eigenvalues can be real <u>or</u> complex
2. Complex eigenvalues are of the form $e^{\pm i\theta}$
   - Outside of stability range, eigenvalues are <u>real</u>

   $$\lambda_a = 1/\lambda_b \qquad \mathrm{Tr}\, M_{RT} = \lambda_a + 1/\lambda_a > 2 \quad \text{if } \lambda_a > 1$$
3. M is not necessarily unitary (where $M^{-1} = M^{\dagger}$)

Stability for *Gaussian* beams in resonators

- A stable resonator mode is one that repeats itself on each round trip
  - Amplitude and phase are matched $\therefore q_{n+1} = q_n$

$$q_1 = \frac{Aq_0 + B}{Cq_0 + D} = q_0 \to Aq_0 + B = q_0\left(Cq_0 + D\right)$$

$$\to 0 = Cq_0{}^2 + \left(D - A\right)q_0 - B$$

$$q_0 = \frac{\left(A - D\right)}{2C} \pm \frac{1}{2C}\sqrt{\left(A - D\right)^2 + 4BC}$$

  - Since $\frac{1}{q_0} = \frac{1}{R} - i\frac{\lambda}{\pi w^2}$ q_0 must be complex (w is finite)

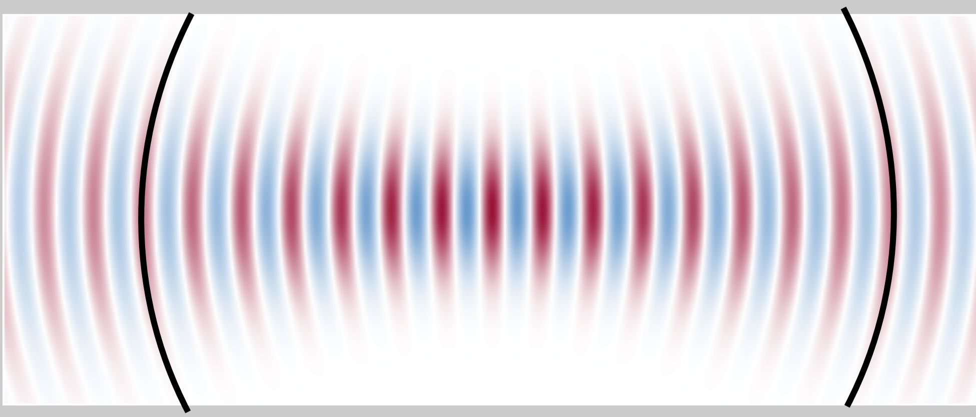
$$\therefore \left(A - D\right)^2 + 4BC < 0$$

Stability for Gaussian beams in resonators

- We know: $(A-D)^2 + 4BC < 0$
- And, since det(M) = 1 $\quad AD - BC = 1$

$$(A-D)^2 + 4BC = (A-D)^2 + 4(AD-1)$$
$$= A^2 - 2AD + D^2 + 4AD - 4$$
$$= (A+D)^2 - 4 < 0$$

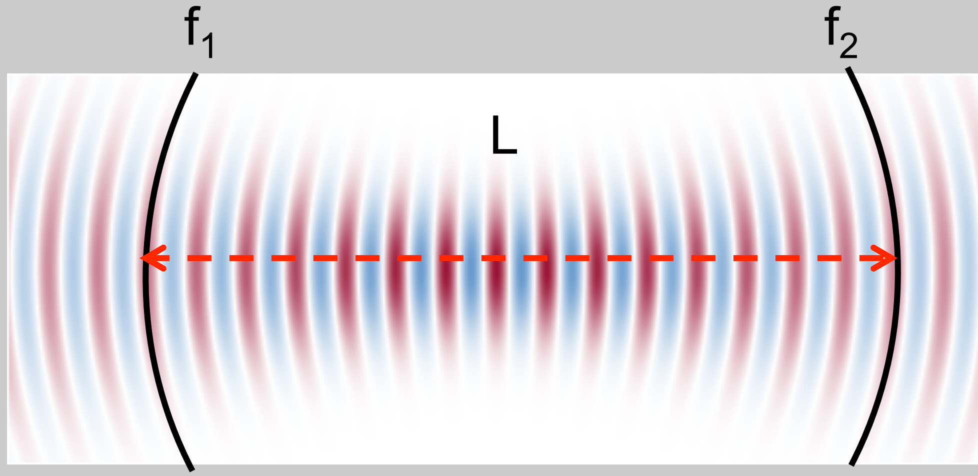
- Stability condition: $\dfrac{(A+D)^2}{4} < 1$



If this condition is satisfied, curvature of each end mirror matches wavefront curvature.

2 mirror cavity stability

- Important example
  - many resonators can be mapped to a 2 mirror cavity



$$M = \begin{pmatrix} 1 & 0 \\ -1/f_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 & L \\ -1/f_1 & 1 - L/f_1 \end{pmatrix} \begin{pmatrix} 1 & L \\ -1/f_2 & 1 - L/f_2 \end{pmatrix}$$

Stability for 2 mirror resonator

- Stability condition: $\frac{(A+D)^2}{4} < 1 \rightarrow -1 < \frac{A+D}{2} < 1$
  - Evaluate A and D from round-trip matrix

$$M = \begin{pmatrix} 1 & L \\ -1/f_1 & 1-L/f_1 \end{pmatrix} \begin{pmatrix} 1 & L \\ -1/f_2 & 1-L/f_2 \end{pmatrix}$$

$$A = 1 - L/f_2$$

$$D = -L/f_1 + (1-L/f_1)(1-L/f_2)$$

$$f_1 = R_1/2 \qquad f_2 = R_2/2$$

$$\frac{A+D}{2} = \frac{1}{2}\left(1 - \frac{2L}{R_2} - \frac{2L}{R_1} + 1 - \frac{2L}{R_1} - \frac{2L}{R_2} + \frac{4L^2}{R_1 R_2}\right)$$

$$= 1 - \frac{2L}{R_1} - \frac{2L}{R_2} + \frac{2L^2}{R_1 R_2} = 2\left(1 - \frac{L}{R_1}\right)\left(1 - \frac{L}{R_2}\right) - 1 \equiv 2 g_1 g_2 - 1$$

2 mirror stability and the stability map

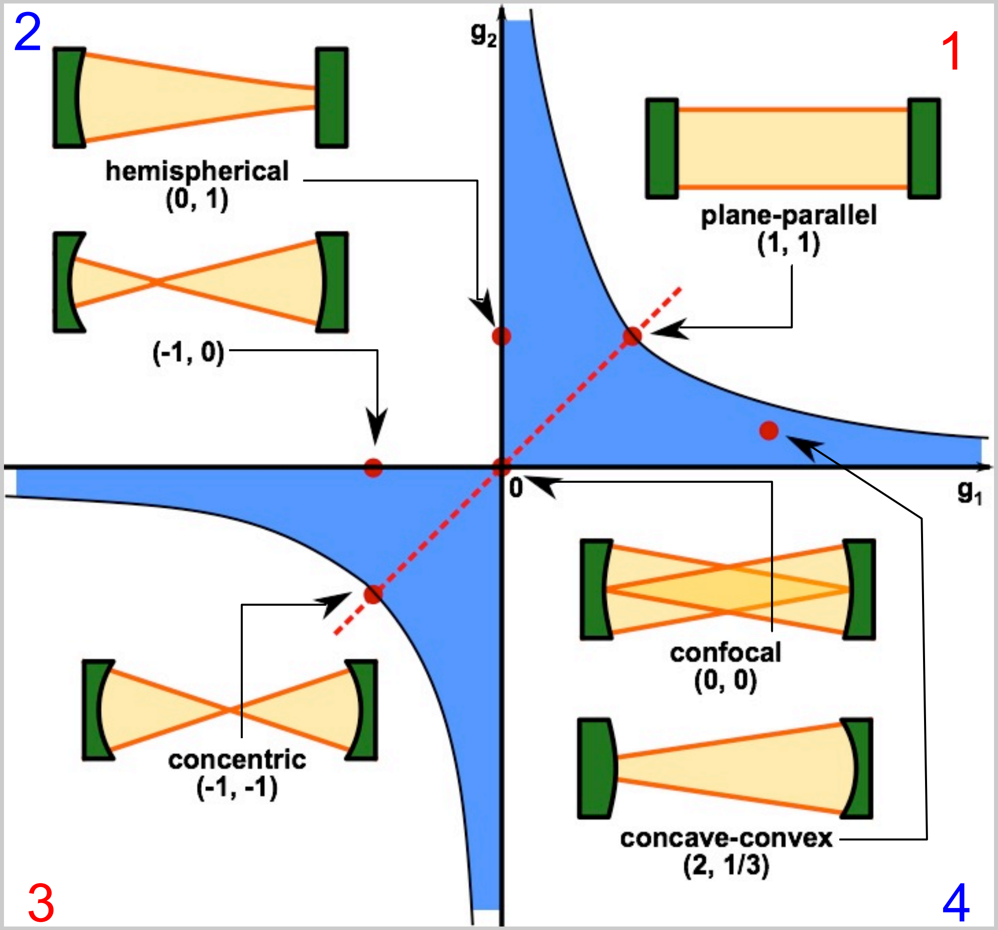
- Cavity is stable if $-1 < \frac{A+D}{2} < 1$

$$-1 < 2g_1 g_2 - 1 < 1$$

$$0 \le g_1 g_2 \le 1$$

$$g_1 = 1 - \frac{L}{R_1} \qquad g_2 = 1 - \frac{L}{R_2}$$

Stable in shaded regions
Unstable in white regions



1st and 3rd quadrants:
Positive branch:
$0 < g_1 g_2 < 1$ **stable**
$g_1 g_2 > 1$ **unstable**

2nd and 4th quadrants:
Negative branch: $g_1 g_2 < 0$
All resonators are unstable in these quadrants

Boundaries of stability

$$g_1 = 1 - \frac{L}{R_1} \quad g_2 = 1 - \frac{L}{R_2}$$

- Easily identified stable resonators are actually at edge of stabilty