6-3. The element of distance in three-dimensional space is

$$dS = \sqrt{dx^2 + dy^2 + dz^2} \tag{1}$$

Suppose x, y, z depends on the parameter t and that the end points are expressed by $\left(x_1(t_1), y_1(t_1), z_1(t_1)\right)$, $\left(x_2(t_2), y_2(t_2), z_2(t_2)\right)$. Then the total distance is

$$S = \int_{t_1}^{t_2} \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2 + \left[\frac{dz}{dt}\right]^2}\, dt \tag{2}$$

The function f is identified as

$$f = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \tag{3}$$

Since $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0$, the Euler equations become

$$\left.\begin{aligned} \frac{d}{dt}\frac{\partial f}{\partial \dot{x}} &= 0 \\ \frac{d}{dt}\frac{\partial f}{\partial \dot{y}} &= 0 \\ \frac{d}{dt}\frac{\partial f}{\partial \dot{z}} &= 0 \end{aligned}\right] \tag{4}$$

from which we have

$$\left.\begin{aligned} \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} &= \text{constant} \equiv C_1 \\ \frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} &= \text{constant} \equiv C_2 \\ \frac{\dot{z}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} &= \text{constant} \equiv C_3 \end{aligned}\right] \tag{5}$$

From the combination of these equations, we have

$$\left.\begin{aligned} \frac{\dot{x}}{C_1} &= \frac{\dot{y}}{C_2} \\ \frac{\dot{y}}{C_2} &= \frac{\dot{z}}{C_3} \end{aligned}\right] \tag{6}$$

If we integrate (6) from t_1 to the arbitrary t, we have

$$\left.\begin{aligned} \frac{x-x_1}{C_1} &= \frac{y-y_1}{C_2} \\ \frac{y-y_1}{C_2} &= \frac{z-z_1}{C_3} \end{aligned}\right] \tag{7}$$

On the other hand, the integration of (6) from t_1 to t_2 gives

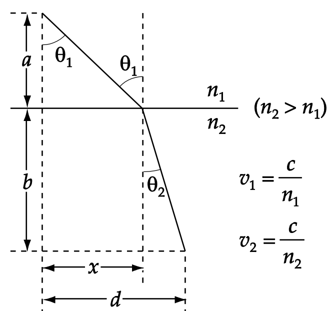
$$\left.\begin{aligned} \frac{x_2-x_1}{C_1} &= \frac{y_2-y_1}{C_2} \\ \frac{y_2-y_1}{C_2} &= \frac{z_2-z_1}{C_3} \end{aligned}\right] \tag{8}$$

from which we find the constants C_1, C_2, and C_3. Substituting these constants into (7), we find

$$\boxed{\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}} \tag{9}$$

This is the equation expressing a straight line in three-dimensional space passing through the two points (x_1, y_1, z_1), (x_2, y_2, z_2).

6-7.



The time to travel the path shown is (cf. Example 6.2)

$$t = \int \frac{ds}{v} = \int \frac{\sqrt{1+y'^2}}{v}\, dx \tag{1}$$

Although we have $v = v(y)$, we only have $dv/dy \neq 0$ when $y = 0$. The Euler equation tells us

$$\frac{d}{dx}\left[\frac{y'}{v\sqrt{1+y'^2}}\right] = 0 \tag{2}$$

Now use $v = c/n$ and $y' = -\tan\theta$ to obtain

$$n \sin\theta = \text{const.} \tag{3}$$

This proves the assertion. Alternatively, Fermat's principle can be proven by the method introduced in the solution of Problem 6-8.

6-12. The path length is given by

$$s = \int ds = \int \sqrt{1 + y'^2 + z'^2}\, dx \tag{1}$$

and our equation of constraint is

$$g(x, y, z) = x^2 + y^2 + z^2 - \rho^2 = 0 \tag{2}$$

The Euler equations with undetermined multipliers (6.69) tell us that

$$\frac{d}{dx}\left[\frac{y'}{\sqrt{1 + y'^2 + z'^2}}\right] = \lambda \frac{dg}{dy} = 2\lambda y \tag{3}$$

with a similar equation for z. Eliminating the factor λ, we obtain

$$\frac{1}{y}\frac{d}{dx}\left[\frac{y'}{\sqrt{1 + y'^2 + z'^2}}\right] - \frac{1}{z}\frac{d}{dx}\left[\frac{z'}{\sqrt{1 + y'^2 + z'^2}}\right] = 0 \tag{4}$$

This simplifies to

$$z\left[y''\left(1 + y'^2 + z'^2\right) - y'\left(y'y'' + z'z''\right)\right] - y\left[z''\left(1 + y'^2 + z'^2\right) - z'\left(y'y'' + z'z''\right)\right] = 0 \tag{5}$$

$$zy'' + \left(yy' + zz'\right)z'y'' - yz'' - \left(yy' + zz'\right)y'z'' = 0 \tag{6}$$

and using the derivative of (2),

$$\left(z - xz'\right)y'' = \left(y - xy'\right)z'' \tag{7}$$

This looks to be in the simplest form we can make it, but is it a plane? Take the equation of a plane passing through the origin:

$$Ax + By = z \tag{8}$$

and make it a differential equation by taking derivatives (giving $A + By' = z'$ and $By'' = z''$) and eliminating the constants. The substitution yields (7) exactly. This confirms that the path must be the intersection of the sphere with a plane passing through the origin, as required.