E. Kreyszig, Advanced Engineering Mathematics, 9th ed.	Section 11.1, pgs. 478-486

Lecture: Fourier Series Module: 09

Suggested Problem Set: {2, 3, 6, 9, 11, 17, 18, 19, 22} Last Compiled : March 4, 2010

E. Kreyszig, Advanced Engineering Mathematics, 9th ed.	Section 11.2, pgs. 487-490

Lecture: FS : Scaling Arguments Module: 09

Suggested Problem Set: {1, 5, 7, 9, 11} Last Compiled : March 4, 2010

E. Kreyszig, Advanced Engineering Mathematics, 9th ed.	Section 11.3, pgs. 490-496

Lecture: FS : Symmetric Functions Module: 09

Suggested Problem Set: {2, 3, 12, 13} Last Compiled : March 4, 2010

Quote of Lecture 9

Professor Hubert Farnsworth: Good news, everyone. You'll be making a delivery to the planet Trisol. A mysterious planet located in the mysterious depths of the Forbidden Zone.

Leela: Professor, are we even allowed in the Forbidden Zone?

Professor Hubert Farnsworth: Why of course. It's just a name, like the Death Zone, or the Zone of No Return. All the zones have names like that in the Galaxy of Terror.

Futurama: My Three Suns (1999)

1. Introduction/Overview

To motivate our study of Fourier methods, I will begin with a question:

- Suppose we are given a function f, which is periodic in the sense that $f(x+p) = f(x)$ for some $p \in \mathbb{R}^+$ and all $x \in \mathbb{R}$. Is there a systematic way to represent f in terms of the periodic functions $\sin(\omega x)$ and $\cos(\omega x)$, whose calculus is well understood?

Naturally, we might also ask if these techniques can be applied to functions that are NOT periodic and lastly what, if anything, does this analysis tell us about *natural phenomena*? As it turns out it will not be difficult to define the systematic process and turn its crank. However, interpretation of the results is non-trivial and will lead to deep mathematical interpretations of physical reality that will be important to the study of signal analysis and partial differential equations. The following will outline some key concepts, terminology and notation, which will be used to build the so-called Fourier series, but first we present one last motivation.

Consider, `http://en.wikipedia.org/wiki/Sawtooth_wave`, which describes the existence and application of a so-called sawtooth wave. This function describes some physical process displaying linear growth on some domain $[-\pi, \pi]$, which is then repeated throughout $\mathbb{R}$. This 'wave-form' is a ubiquitous starting point for various signal analysis. As it turns out the quick question,

- What are the derivative and integral properties of the sawtooth function?

is not a straight-forward one. The functions graph is simply understood, however, it does not admit any clear derivatives or integrals. So, the first focal point of Fourier analysis will be taking this function and expressing it as the infinite sum of simpler periodic functions,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}2}{n} \sin(nx), \tag{1}$$

which we can differentiate and integrate with little fear. First, we explain how to exploit the **orthogonality of the basis vectors** in order to derive this expression. After this we study extensions of this method to any periodic function and discuss simplifications of the method due to symmetry. The latter will lead to the esoteric statement,

- Any 'arbitrary' function $f : \mathbb{R} \to \mathbb{R}$ whose domain is a bounded subset of the real-line has a representation as a trigonometric series.[1]

which along with its non-periodic analogs will be of considerable importance in the study of linear PDE. [2]

2. Review Material - Periodic Functions

The following content is meant to be a review. If it is not then please study it and ask me questions if there are statements, which still appear unclear to you.

Definition 1. *Periodic Function*
We say that the function $f : D \to \mathbb{R}$ is a periodic function if $D \subset \mathbb{R}$ is a bounded set and $f(x+p) = f(x)$ is true for all $x \in \mathbb{R}$ and some positive p. We call p a period of f and the smallest such p is called the principle period of f.

Remark 1. *So, it is natural to ask, what is p? Well, consider the following functions*

$$f_1(x) = \cos(x), \tag{2}$$

$$f_2(x) = \sin(x), \tag{3}$$

and notice that each function is such that $f_1(x+4\pi) = f_1(x)$ and $f_2(x+4\pi) = f_2(x)$. However, this is not the smallest p one can find. If we instead consider $p = 2\pi$ we have the similar conclusion that $f_1(x+2\pi) = f_1(x)$ and $f_2(x+2\pi) = f_2(x)$. In fact this is the smallest such p such that $f(x+p) = f(x)$, which we known because the principle domain of each function is the 2π interval from $[-\pi, \pi]$. So, we note,

- *The principle period of a periodic function is the length of the smallest interval needed to recreate the rest of the functions graph.*

and these examples motivate the following claim.

Claim 1. *Periods of a Periodic Function*
If a periodic function f and principle period p, then it also has periods $\{p, 2p, 3p, \ldots, np\}$ where $n \in \mathbb{N}$.

Proof. See proof in 11.1 of the text. □

Claim 2. *Linear combinations of Periodic Functions*
If f and g are periodic functions with principle period p then h defined by $h(x) = \alpha f(x) + \beta g(x)$ is also a periodic function of period p.

Proof. Notice

$$h(x+p) = \alpha f(x+p) + \beta g(x+p) = \alpha f(x) + \beta g(x) = h(x), \tag{4}$$

for all x and $\alpha, \beta \in \mathbb{R}$. □

Remark 2. *What this says is that arbitrary linear combinations of periodic functions are also periodic.*

Claim 3. *Period Scaling*
Let $g(x) = \sin\left(\frac{n\pi}{L}x\right)$ where $n \in \mathbb{N}$ and $L \in \mathbb{R}^+$, then the principle period of g is $\frac{2L}{n}$.

Proof. We know that sine is a 2π periodic function so we ask, for what x does $\frac{n\pi}{L}x = 2\pi$?Clearly, this is $x = \frac{2L}{n}$ and at this point g must repeat itself. Since 2π is the principle period of sine $\frac{2L}{n}$ is the principle period of g. □

[1] We will later call these trigonometric series Fourier series. In our study we will not focus on the exact meaning of 'arbitrary' since, for almost all physical cases, the Fourier series is well-defined.

[2] We will not concern ourselves with the arbitrariness of f. This is an interesting point but its treatment would require a discussion of the formal convergence of the series and also exactly what we take *integral* to mean.

Remark 3. *It is important to note that since g is $\frac{2L}{n}$-periodic it also has integer-multiple periods of this period. Thus, g is also a $2L$ periodic function. Since similar results hold for the cosine function we can conclude that,*

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(\omega_n x) + b_n \sin(\omega_n x), \tag{5}$$

where $\omega = \frac{n\pi}{L}$ is a $2L$-periodic function. [3]

Remark 4. *Also of importance are the following topics found in the handouts on* `ticc.mines.edu`*:*

- *The unit circle and values for f_1 and f_2 for special values $x \in \left\{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi\right\}$.*
- *Half-angle, angle-sum and various other trigonometric identities.*

3. Review Material - Symmetric Functions

Definition 2. *Even and Odd Functions*
A function f is even if $f(-x) = f(x)$ for all $x \in \mathbb{R}$. A function g is odd if $g(-x) = -g(x)$ for all $x \in \mathbb{R}$. In both cases we say that the function has a symmetry or is symmetric.

Remark 5. *Common examples are $f_{even}(x) = x^2$ and $g_{odd}(x) = x^3$ and explains where the terminology of even and odd comes from.*

Claim 4. *Linear combinations of Symmetric Functions*
Arbitrary linear combinations of symmetric functions are symmetric with the same symmetry of the combined functions.

Proof. Let f and g be even functions, then for $h(x) = f(x) + g(x)$ we have, $h(-x) = f(-x) + g(-x) = f(x) + g(x) = h(x)$, which shows that h is even. The proof for odd functions is similar. □

Remark 6. *Linear combinations of functions with different symmetries have no symmetry.*

Claim 5. *Symmetry of Trigonometric Functions*
The functions cosine and sine are even and odd respectively.

Proof. Appeal to the Taylor series representations of each function from this we see that the terms of the cosine series are even functions while the terms of the sine series are odd functions. □

Claim 6. *Products of Symmetric Functions*
Suppose f and g are symmetric functions and define $h(x) = f(x)g(x)$.

- *If f and g share the same symmetry then h is even.*
- *If f and g have different symmetries then h is odd.*

Proof. Let f and g be odd functions then $h(-x) = f(-x)g(-x) = (-f(x))(-g(x)) = h(x)$. The proof for when f and g are even is similar. Assuming that f is even and g is odd gives $h(-x) = f(-x)g(-x) = f(x)(-g(x)) = -h(x)$. □

Claim 7. *Integral Properties of Symmetric Functions*
Let f be an even and let g be an odd function then for some $a \in \mathbb{R}^+$ we have,

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx, \tag{6}$$

$$\int_{-a}^{a} g(x)dx = 0. \tag{7}$$

Proof. The previous equalities follow from basic integral properties and u-substitution.

$$\begin{aligned}
\int_{-a}^{a} f(x)dx &= \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx = -\int_{a}^{0} f(-u)du + \int_{0}^{a} f(x)dx = \int_{0}^{a} f(u)du + \int_{0}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx, \\
\int_{-a}^{a} g(x)dx &= \int_{-a}^{0} g(x)dx + \int_{0}^{a} g(x)dx = -\int_{a}^{0} g(-u)du + \int_{0}^{a} g(x)dx = -\int_{0}^{a} g(u)du + \int_{0}^{a} g(x)dx = 0
\end{aligned}$$

□

[3] Notice that $a_0 \in \mathbb{R}$ repeats itself for every p.

4. Lecture Goals

Our goals with this material will be:

- Construct a system for representing periodic function in terms of the since/cosine basis.
- Understand how Fourier series can be interpreted in terms of linear combinations and inner-products.
- Interpret the meaning of the data produced by Fourier methods in terms of amplitudes of oscillatory functions.
- Extend the idea of Fourier series to functions of any period and symmetric functions.

5. Lecture Objectives

The objectives of these lessons will be:

- Learn to calculate the Fourier coefficients given a specific periodic function.
- Write down and graph Fourier series representations and approximations of periodic functions.
- Introduce the concept of half-range expansions as they relate to Fourier sine/consine series and functions with symmetry.