PHGN 462 Homework 13

1) The dipole radiation fields that we obtained in class are pretty complicated, but they become fairly tidy (and still useful) in certain limits. Let's get a little practice working with those fields, and do a simple lead-in to the idea of interference.

a) Assume we have an electric dipole sitting at the origin that varies in time harmonically (a so-called Hertzian dipole) with the dipole moment pointing in the $\hat{k}$ direction. Write down the radiative E-field associated with such a dipole.

b) Now let's suppose we have two dipoles of the same frequency oscillating in phase with one another and separated by some distance d along the y axis – in other words, we have one source at $(0, -\frac{d}{2}, 0)$ and another at $(0, \frac{d}{2}, 0)$. These two sources will interfere with one another. In some places, their radiative fields will add constructively, and in others they'll add destructively, leading to a characteristic pattern.

Let's further restrict ourselves to the equatorial plane, $\theta = \frac{\pi}{2}$, and look at a snapshot of the radiation at $t = 0$. Show formally that if we're looking some decent distance away from the sources, the magnitude of the E-field at any point will be given by:

$$E = \frac{e^{ikr_1} + e^{ikr_2}}{r}$$

where r is the distance from the origin to the observation point, and r_1 and r_2 are the distances from the individual sources to the observation point.

c) Show that the radiation from this double dipole will be concentrated along certain directions, with maxima sitting at angles that satisfy

$$\sin \varphi = \frac{n\lambda}{d}$$

where n is some integer, and φ is the usual spherical coordinate (the angle off the x-axis towards the y-axis).

Note that at this point we've reproduced the double-slit interference condition from Physics 200, except that this time our argument traces an explicit, unbroken line all the way back to the Maxwell equations.

d) Time for some funsies. In Mathematica (or whatever suits you), make a 2D plot that shows off these interference-based beams of radiation. I like using DensityPlot, but whatever floats your boat. You might have some dynamic range issues… the field values involved span enough orders of

magnitude that Mathematica's default color mapping will probably wash out the detail. You may have already figured out how to deal with that back in the Gaussian beams assignment, so do whatever works for you.

Once you have that all set up, show (either with a plot or analytically) that you can steer the beams by changing the phase of one of the sources relative to the other. I like using Mathematica's Manipulate feature to get a controllable animation, but again, whatever floats your boat.

That last bit is really important physically, by the way. You can make steerable beams of radiation simply by constructing an array of dipole transmitters and adjusting their relative phase. These so-called *phased arrays* are controlled strictly electronically and have no moving parts. They're lightning quick, versatile, and super robust, and therefore get a lot of use in the modern world.

2) Let's suppose we have a whole bunch of monochromatic plane waves of different but equally-spaced frequencies, all relatively in phase with each other, all of the same polarization, all of the same field amplitude, all traveling in the same direction. What I mean by that is that each plane wave has an electric field of the form

$$E_n = E_0 e^{-i(\omega_n t + \delta_n)}$$

where all the δ_n are zero, and each ω_n is $\Delta\omega$ more than the previous, so that we can write

$$\omega_n = \omega + n\Delta\omega$$

a) Show that if you superimpose N of these waves, you'll get a series of sharp pulses. In other words, find intensity as a function of time for that superposition and interpret it.

b) Come up with an expression for the width of one of the pulses in terms of given quantities. This is intentionally vague since there's more than one way to define the width of a pulse. Do what you like, as long as you end up with some kind of analytic expression so that you can tell me how the pulse width depends on the total number of constituent waves N and the frequency separation $\Delta\omega$.

For your reference, laser cavities tend to naturally produce lots of modes of different-but-equally-spaced frequencies. This is hinted at by our upcoming treatment of Fabry-Perot etalons. Generating a bunch of said modes and then somehow arranging for them to all be in phase with one another is how people make really, *really* short-pulse lasers these days – it's called mode locking.

3) In this one, we're going to come up with and apply a procedure for recovering the index of refraction of a material from that material's absorption spectrum. This is the real stuff, so as with all stuff that you can actually use, there may be some annoying, fiddly bits. But it'll be worth it.

a) First off, let's assume that we have a material that is both refractive and absorptive. That means that we'll be able to write all sorts of stuff, from the permittivity to the susceptibility, as complex-valued functions of wavelength. The real parts represent the refractive behavior, and the imaginary parts represent that absorption.

Let the susceptibility of the material be:

$$\chi = \chi_1 + i\chi_2$$

Working from the definition of index of refraction and relationships between index, permittivity, and susceptibility, show that if the susceptibility is small, we can write the real part of the index of refraction as:

$$\text{Re}\{n\} \approx 1 + \frac{\chi_1}{2}$$

Keep in mind that all of these things are functions of frequency ω.

b) So how the heck do we actually measure χ_1? Well, we could send light of lots of different colors through the material and see how much each color refracts, thus obtaining n and χ_1. But that's hard, and we don't usually do that. Instead we measure absorption, and use the Kramers-Kronig relations to get from absorption to what we really want.

Define the absorption function $\delta(\omega)$ to be the reciprocal of the absorption length d, which itself is defined on p. 513 of Pollack and Stump:

$$\delta(\omega) = d^{-1}$$

Working from the basic dispersion relation for plane waves, and from various identities linking the wavevector, permittivity, and susceptibility, show that:

$$\delta(\omega) \approx \frac{\omega\chi_2}{c}$$

c) Use the above results and the alternate form of the Kramers-Kronig relations (the form with the domain that goes from zero to infinity) to show that the real part of the index of refraction is related to the absorption spectrum via:

$$Re\{n(\omega)\} = 1 + \frac{c}{\pi}\int_0^\infty \frac{\delta(\omega')}{(\omega'^2 - \omega^2)}d\omega'$$

Pay close attention to where the omegas and omega primes are, as they contain an important bit of physics: To get the index at *any* frequency, you need to know about the absorption at *all* frequencies.

d) Now the fun part: Let's say we have a material that is strongly absorptive at one particular frequency, and not terribly absorptive anywhere else. Write down a function $\delta(\omega)$ that models that behavior. Then use the relationship from (c) to generate a plot of $Re\{n(\omega)\}$ for that material. This will almost certainly need to be done numerically. I'm only concerned with finding qualitative behavior, so feel free to set constants to one and not worry about units.

Depending on what kind of $\delta(\omega)$ you use, the integral may or may not converge cleanly. This is where it can get fiddly. Mathematica has a whole host of options for tweaking its integration routine, and the latest version is decent about recommending them to you based on the manner in which the integration is failing to converge. You can also try different $\delta(\omega)$ functions and see if some integrate better than others (hint: some do).

e) Look at your plot and tell me whether or not it has the 'expected' pattern of normal and anomalous dispersion. I'm actually not certain whether *all* absorption functions necessarily generate anomalous dispersion in the vicinity of an absorption peak (but they usually do), so let's see what you get. I'm not looking for a whole lot of discussion on this part; feel free to be brief.

4) Third peer lecture, as discussed in class (due Friday December 4th – not same day as rest of assignment)