1_30_08

Note Title 1/29/2008

we show that for QHO

$$E \geq \frac{\hbar\omega}{2}$$

Hence there is a lowest energy state: the ground state ψ_0

Now suppose we assert that

$$a\psi_0 = 0 \implies \frac{d\psi_0}{dx} + \frac{m\omega}{\hbar} x \psi_0 = 0$$

$$\implies \psi_0 = A e^{-\frac{m\omega}{2\hbar} x^2}$$

<u>Claim</u>: this is the ground state

proof: $H\psi_0 =$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_0 + \frac{1}{2}m\omega^2 x^2 \psi_0$$

$$= A\left[-\frac{\hbar^2}{2m}\left(-\frac{m\omega}{\hbar} + \left(\frac{m\omega}{\hbar}\right)^2 x^2\right) e^{-m\omega/2\hbar\, x^2}\right]$$

$$+ A\frac{1}{2}m\omega^2 x^2 e^{-m\omega/2\hbar\, x^2}\Big]$$

$$= A\left\{\frac{\hbar\omega}{2} - \frac{1}{2}m\omega^2 x^2 + \frac{1}{2}m\omega^2 x^2\right\} e^{-\frac{m\omega}{2\hbar}x^2}$$

$$= \frac{\hbar\omega}{2} A e^{-\frac{m\omega}{2\hbar}x^2} = \frac{\hbar\omega}{2}\psi_0$$

so $\quad H\psi_0 = \frac{\hbar\omega}{2}\psi_0$

Thus ψ_0 is an ε-vector of H with ε-value $\frac{\hbar\omega}{2}$. Since

$E \geq \frac{\hbar\omega}{2}$ this must be the

ground state.

Example $\quad A_1\psi_1 = a^+\psi_0$

$\uparrow$

normalization

$$a^+\psi_0 = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{i}{m\omega}\left(-i\hbar\frac{d}{dx}\right)\right)\psi_0$$

$$= \sqrt{\frac{m\omega}{2\hbar}}\left[x - \frac{\hbar}{m\omega}\frac{d}{dx}\right]\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega/2\hbar\, x^2}$$

$$= \sqrt{\frac{m\omega}{2\hbar}} \left[x + \frac{\hbar}{m\omega} \frac{m\omega}{\hbar} x \right] \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega/2\hbar \, x^2}$$

$$= 2\sqrt{\frac{m\omega}{2\hbar}} \; x \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega/2\hbar \, x^2}$$

$$\sqrt{2} \; \pi^{-1/4} \left(\frac{m\omega}{\hbar}\right)^{3/4} x \, e^{-m\omega/2\hbar \, x^2}$$

Normalize it:

$$\psi_1(x) = \sqrt{2} \; \pi^{-1/4} \alpha^{3/4} \; x \; e^{-\alpha/2 x^2} \qquad \alpha = \frac{m\omega}{\hbar}$$

or $$\boxed{\psi_1(x) = \sqrt{2} \left(\frac{\alpha}{\pi}\right)^{1/4} y \; e^{-y^2/2}}$$

Simplify notation

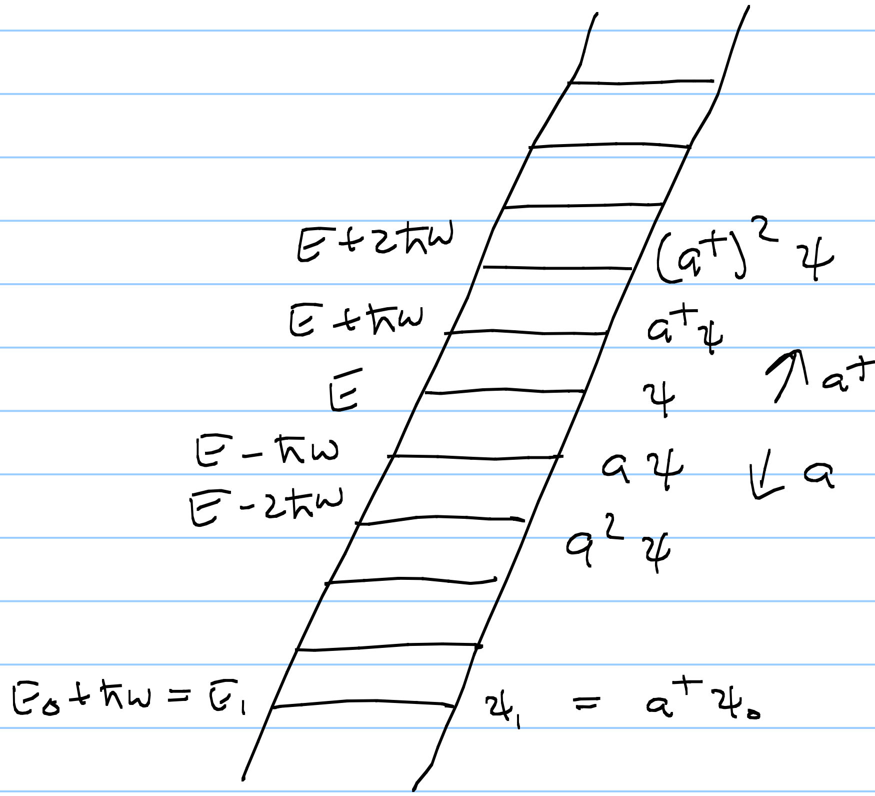
$$e^{-m\omega/2\hbar \, x^2} = e^{-y^2}$$

$$y = \sqrt{\alpha} \, x \qquad \boxed{\alpha = \frac{m\omega}{\hbar}}$$

$$\psi_0 = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-y^2/2}$$

$$\frac{d}{dx} = \sqrt{\alpha} \frac{d}{dy}$$

Ladder operators for QHO



All we need is on stationary state and we can compute all the rest by raising with a^+ or Lowering with a.

$$\boxed{H\psi_N = \hbar\omega\left(N+\tfrac{1}{2}\right)\psi_N} \quad \text{QHO}$$

Review continued.

QHO. $\omega^2 = \frac{k}{m} \Rightarrow k = m\omega^2$

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$$

Be able to prove that $\boxed{[x, p] = i\hbar}$

Ladder operators

$$a = \frac{m\omega}{2\hbar}\left(x + \frac{i}{m\omega} p\right)$$

$$a^\dagger = \frac{m\omega}{2\hbar}\left(x - \frac{i}{m\omega} p\right)$$

Eg $a\psi_E$

use $[H, a] = -\hbar\omega a$

i.e. $Ha - aH = -\hbar\omega a$

$$\Rightarrow H a\psi_E = a \underbrace{H\psi_E}_{= E\psi_E} - \hbar\omega a \psi_E$$

since its an eigenstate

$$\Rightarrow H a\psi_E = E a\psi_E - \hbar\omega a\psi_E$$

$$\Rightarrow H[a\psi_E] = E - \hbar\omega [a\psi_E]$$

So $a\psi_E$ is an ε-state of H with ε-value $E-\hbar\omega$: Lowering

Similarly using $[H, a^+] = \hbar\omega a^+$

$\Rightarrow \quad H\,[a^+\psi_E] = (E+\hbar\omega)\,[a^+\psi_E]$

So $a^+\psi_E$ is an ε-state of H with ε-value $E+\hbar\omega$: raising.

Summary of what we know so far.

QM (non-relativistic anyway) is governed by TDSE

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi$$

S. of Var. $\Psi(x,t) = \psi(x)\varphi(t)$

gives $\boxed{\Psi(x,t) = \underbrace{\psi(x)}\, e^{-iEt/\hbar}}$

analagous to normal modes in classical physics

Equation for $\psi(x)$ is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

or

$$\underbrace{\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]}_{\text{Hamiltonian operator}}\psi = E\psi$$

$$\boxed{H\psi = E\psi}$$

So the spatial part of the S. of V. is an ε-value ε-vector problem for H.

H is self-adjoint (Hermitian)

So 1) H has real ε-values
2) a complete set of orthog. ε-vectors

$$H \psi_n = E_n \psi_n$$

$$\Psi_n(x,t) = \psi_n e^{-iE_n t/\hbar}$$ Solution to TDSE

But $\Psi_n(x,0) = \psi_n(x)$ not general enough for arbitrary B.C. So we exploit linearity of S.E.

$$\Psi(x,t) = \sum c_n \psi_n(x) e^{-iE_n t/\hbar}$$

to calculate c_n we do

$$\Psi(x,0) = \sum c_n \psi_n(x)$$

$$\int_{-\infty}^{\infty} \psi_m^*(x)\, \Psi(x,0)\, dx = \sum_n c_n \underbrace{\int \psi_m^* \psi_n \, dx}_{\delta_{mn}}$$

$$= c_m$$

$$\boxed{c_m = \int_{-\infty}^{\infty} \psi_m^*(x)\, \Psi(x,0)\, dx}$$

E.g. for the ∞- square well

$$c_m = \sqrt{\frac{2}{a}} \int_{-\infty}^{\infty} \sin\left(\frac{n\pi x}{a}\right) \Psi(x,0)\, dx$$

$|c_n|^2$ tells you the probability that a measurement of the energy would yield a value E_n.

Hence the sum of the $|c_n|^2$ must be 1.

$$\sum_{n=1}^{\infty} |c_n|^2 = 1$$

Proof
$$1 = \int_{-\infty}^{\infty} |\Psi(x,0)|^2 dx$$

$$= \int_{-\infty}^{\infty} \left(\sum_m c_m \psi_m\right)^* \left(\sum_n c_n \psi_n\right)$$

$$= \int \sum_m \sum_n c_m^* c_n \psi_m^* \psi_n \, dx$$

$$= \sum_m \sum_n c_m^* c_n \underbrace{\int \psi_m^* \psi_n \, dx}_{\delta_{mn}}$$

$$= \sum c_m^* c_m = \sum |c_m|^2 = 1$$

$$\langle H \rangle = \int \Psi^*(x,t) \, H \, \Psi(x,t) \, dx$$

$$\Psi(x,t) = \sum c_n \psi_n(x) e^{-iE_n t/\hbar}$$

$$\langle H \rangle = \int \sum_n c_n^* \psi_n^* e^{i(\ )} \, H \sum_m c_m \psi_m e^{-i(\ )}$$

$$= \int \sum_N \sum_m c_N^* \psi_N^* \underline{e^{i()}}\, c_m E_m \psi_m \underline{e^{-i()}}$$

cancel

$$= \sum_N \sum_m c_N^* c_m E_m \int \psi_N^* \psi_m \, dx$$

$$\boxed{\langle H \rangle = \sum_N E_N |c_N|^2}$$

This in independent of time

manifestation of conservation of energy.

we did example 2.2 last time
∞ square well with

$$\Psi(x,0) = Ax(a-x) \quad 0 \le x \le a$$

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)$$

we computed

$$c_n = \begin{cases} 0 & n \text{ even} \\ \frac{8\sqrt{15}}{(n\pi)^3} & n \text{ odd} \end{cases}$$

So Let's verify:

$$|c_1|^2 = \left(\frac{8\sqrt{15}}{\pi^3}\right)^2 = .99855$$

So a measurement of Energy has a 99.855 % prob. of yielding
$E_1 = \pi^2\hbar^2/2ma^2$

less obvious

$$\sum_{n=1}^{\infty} |c_n|^2 = \left(\frac{8\sqrt{15}}{\pi^3}\right)^2 \sum_{n=1,3,5} \frac{1}{n^6} \equiv \underline{1}$$

of course I wouldn't expect you to know that

$$\Sigma = 1 + \left(\tfrac{1}{3}\right)^6 + \left(\tfrac{1}{5}\right)^6 + \cdots = \frac{\pi^6}{960}$$

but $\left(\tfrac{1}{5}\right)^6 = \dfrac{1}{15625}$ a very smal number.

so to a good approx $\Sigma \cong 1+\left(\tfrac{1}{3}\right)^6$

$= 1.00137$

Review continued.