

Eigenvalues - Eigenvectors - Diagonalization - Spectral Decomposition - Applications

1. Given,

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}.$$

- (a) Determine the eigenvalues of \mathbf{A} .
- (b) Determine the eigenvectors of \mathbf{A} .

2. Given,

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}.$$

Determine the eigenvalues and eigenfunctions associated with the system of differential equations $\frac{dx}{dt} = \mathbf{A} \cdot \mathbf{x}(t)$.

3. Given,

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$$

If \mathbf{A} is diagonalizable, then determine \mathbf{D} and \mathbf{P} associated with its decomposition \mathbf{PDP}^{-1} . Do not find \mathbf{P}^{-1} .

4. Square matrices having columns whose entries sum to 1 are often called stochastic matrices. Those with only non-negative entries, for some power, are called *regular* stochastic matrices. Given a random process, with an initial state \mathbf{x}_0 , the application of \mathbf{P} on \mathbf{x}_0 discretely steps the process forward in time. That is $\mathbf{x}_{n+1} = \mathbf{P}\mathbf{x}_n = \mathbf{P}^n\mathbf{x}_0$, $n = 1, 2, 3, \dots$. If a matrix is a *regular* stochastic matrix then there exists a steady-state vector \mathbf{q} such that $\mathbf{P}\mathbf{q}=\mathbf{q}$. This vector determines the long term probabilities associated with an arbitrary initial state \mathbf{x}_0 . The sequence of states, $\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{n+1}\}$, is called a *Markov Chain*. Given the regular stochastic matrix:

$$\mathbf{P} = \begin{bmatrix} .1 & .6 \\ .9 & .4 \end{bmatrix}.$$

- (a) Show that the steady-state vector of \mathbf{P} is $\mathbf{q} = \left[\frac{2}{5} \quad \frac{3}{5} \right]^T$.
- (b) Find the matrices \mathbf{D} and \mathbf{Q} such that $\mathbf{P} = \mathbf{QDQ}^{-1}$. That is, diagonalize the matrix \mathbf{P} .
- (c) Show that $\lim_{n \rightarrow \infty} \mathbf{P}^n \mathbf{x}_0 = \mathbf{q}$ where $\mathbf{x}_0 = [x_1, x_2]^T$ is an arbitrary vector in \mathbb{R}^2 such that $x_1 + x_2 = 1$.

5. Recall the Pauli Spin Matrix from homework 1,

$$\sigma_2 = \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

- (a) Show that σ_y is self-adjoint.
- (b) Find the orthogonal diagonalization of σ_y .
- (c) Show that $\sigma_y = \lambda_1 \mathbf{x}_1 \mathbf{x}_1^H + \lambda_2 \mathbf{x}_2 \mathbf{x}_2^H$, where \mathbf{x}_1 and \mathbf{x}_2 are the normalized eigenvectors from part (b).