MATH348 - March 1, 2010
Exam I - 50 Points - 50 minutes

NAME: Key
SECTION:

In order to receive full credit, SHOW ALL YOUR WORK. Full credit will be given only if all reasoning and work is provided. When applicable, please enclose your final answers in boxes.

1. (10 Points)

(a) True/False : Mark each statement as either true or false.

i. Suppose that $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} \in \mathbb{R}^{n\times n}$, has no solutions. The corresponding homogeneous system, $\mathbf{Ax} = \mathbf{0}$, has only the trivial, $\mathbf{x} = \mathbf{0}$, solution.

False

ii. If $\mathbf{A} \in \mathbb{R}^{m\times n}$ has a row of zeros then $\mathbf{Ax} = \mathbf{0}$ always has infinitely-many solutions.

False

iii. It is impossible for a vector to be in both the null-space and column-space of a matrix.

False

iv. If the dimension of the column-space of $\mathbf{A}_{n\times n}$ is n then $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution.

True

v. The system $\mathbf{Ax} = \mathbf{0}$, where $\mathbf{A} \in \mathbb{R}^{3\times 4}$ has only the trivial solution.

False

(b) Short Response : Provide a short justification of your conclusion.

i. Suppose $\mathbf{V}_{n\times n}$ is a matrix whose columns form a basis for $\mathbb{R}^n$. What can be said about the determinant of $\mathbf{V}$?

Columns of V form a basis for $\mathbb{R}^n$ $\Rightarrow V \sim I \Rightarrow \det(V) \neq 0$

ii. Suppose that $\lambda = 0$ is an eigenvalue of $\mathbf{A}$. What can be said about $\mathbf{A}^{-1}$?

$\det(A - \lambda I) = \det(A) = 0 \Rightarrow A^{-1}$ does not exist.

or

$A\bar{x} = \lambda\bar{x} = 0\cdot\bar{x} \Rightarrow \bar{x}$ is a nontrivial sol$\underline{n}$ to $A_{nn} \Rightarrow A^{-1}$ DNE

2. (10 Points) Quickies

(a) Given,

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 3 & h & k \end{array}\right]. \sim \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & h-9 & k-6 \end{array}\right]$$

Determine all values of h and k so that the system has:

i. Exactly one solution

$h \neq 9$, $k \in \mathbb{R}$

ii. Infinitely-many solutions

$h=9$, $k=6$

iii. No solutions

$h=9$, $k \neq 6$

(b) Find all eigenvalues of,

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \quad A-\lambda I = \begin{bmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = (4-\lambda)(1-\lambda)^2 + 1\cdot +2(1-\lambda) = 0$$
$$= (1-\lambda)\left[(4-\lambda)(1-\lambda)+2\right] =$$
$$= (1-\lambda)\left[\lambda^2-5\lambda+6\right] = (1-\lambda)(2-\lambda)(3-\lambda) = 6$$
$$\Rightarrow \quad \lambda_1 = 1, \ \lambda_2 = 2, \ \lambda_3 = 3$$

(c) Find all values of h so that the following vectors are linearly independent.

$$\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} -5 \\ 7 \\ 8 \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} 1 \\ 1 \\ h \end{bmatrix}$$

$$\begin{bmatrix} 1 & -5 & 1 \\ -1 & 7 & 1 \\ -3 & 8 & h \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 1 \\ 0 & 2 & 2 \\ 0 & -7 & h+3 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 1 \\ 0 & 1 & 1 \\ 0 & -7 & h+3 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & h+10 \end{bmatrix}$$

$$\Rightarrow h \neq -10$$

(d) Find the general solution to the following linear system of equations.

$$x_2 + 2x_3 = 0$$
$$4x_1 + 5x_2 + 6x_3 = 0$$
$$8x_1 + 9x_2 + 10x_3 = 0$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 4 & 5 & 6 & 0 \\ 8 & 9 & 10 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 4 & 5 & 6 & 0 \\ 8 & 9 & 10 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|c} 4 & 5 & 6 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 4 & 5 & 6 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_2 = -2x_3 \\ 4x_1 = -5x_2 - 6x_3 = -5(-2x_3) - 6x_3 = 4x_3 \end{array}$$

$$\Rightarrow \vec{x} = \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad x_3 \in \mathbb{R}$$

3. (10 Points) Find a basis for the null-space and column-space of,

$$\mathbf{A} = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$B_{Col(A)} = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$\Rightarrow x_4 = -3x_5$$

$$3x_3 = \cancel{4}\, x_4 - x_5 = -3x_5 - x_5 = -4x_5 \Rightarrow x_3 = -\frac{4}{3}x_5$$

$$2x_1 = 3x_2 - 6x_3 - 2x_4 - 5x_5 = 3x_2 + 8x_5 + 6x_5 - 5x_5 =$$

$$= 3x_2 + 9x_5 \Rightarrow$$

$$\Rightarrow \vec{x} = \begin{bmatrix} \frac{3}{2}x_2 + \frac{9}{2}x_5 \\ x_2 \\ -\frac{4}{3}x_5 \\ -3x_5 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 9/2 \\ 0 \\ -4/3 \\ -3 \\ 1 \end{bmatrix} x_5, \quad x_2, x_3 \in \mathbb{R}$$

$$B_{Nul(A)} = \left\{ \begin{bmatrix} 3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 9/2 \\ 0 \\ -4/3 \\ -3 \\ 1 \end{bmatrix} \right\}$$

4. (10 Points) Let $\mathbf{A} = \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix}$.

(a) Find the eigenvalues of A.

$$\det(A - \lambda I) = \left(\frac{3}{4} - \lambda\right)^2 - \frac{1}{16} = \lambda^2 - \frac{3}{2}\lambda + \frac{9}{16} - \frac{1}{16} = \lambda^2 - \frac{3}{2}\lambda + \frac{1}{2} = 0$$

$$\lambda = \frac{\frac{3}{2} \pm \sqrt{9/4 - 4(1)(\frac{1}{2})}}{2} = \frac{\frac{3}{2} \pm \sqrt{\frac{9}{4} - \frac{8}{4}}}{2} = \frac{\frac{3}{2} \pm \sqrt{\frac{1}{4}}}{2} = \frac{\frac{3}{2} \pm \frac{1}{2}}{2} = 1, \frac{1}{2}$$

(b) Find the eigenvectors of A.

$$\lambda_1 = 1 \Rightarrow [A - \lambda_1 I | 0] = \left[\begin{array}{cc|c} -\frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & -\frac{1}{4} & 0 \end{array}\right] \Rightarrow x_1 = x_2 \Rightarrow x^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = \frac{1}{2} \Rightarrow \left[\begin{array}{cc|c} \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 \end{array}\right] \Rightarrow x^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \end{array}\right]$$

(c) Calculate $\lim_{n\to\infty} \mathbf{A}^n$.

$$\lim_{n\to\infty} A^n = \lim_{n\to\infty} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1^n & 0 \\ 0 & (\frac{1}{2})^n \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

5. (10 Points) Suppose that $\mathbf{A}$ has the following eigenvalue, eigenvector pairs.

$$\lambda_1 = 0, \quad \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 1, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_3 = -1, \quad \mathbf{e}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Find the general solution to $\mathbf{Ax} = \mathbf{0}$.

1) $$A\vec{x} = A(c_1\vec{e}_1 + c_2\vec{e}_2 + c_3\vec{e}_3) = c_1\lambda_1\vec{e}_1 + c_2\lambda_2\vec{e}_2 + c_3\lambda_3 e_3 =$$
$$= c_2 e_2 + c_3 e_3 = 0 \iff e_2 \cdot (c_2 e_2 + c_3 e_3) = e_2 \cdot 0$$
$$\Rightarrow \quad c_2 = 0$$
same for e_3.
$$\Rightarrow \quad c_1 \in \mathbb{R}$$

2) $$A\vec{x} = PDP^{-1}\vec{x} = \vec{0} \iff D\tilde{x} = P^{-1}\vec{0} = \vec{0} \iff \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\Rightarrow \tilde{x}_2 = \tilde{x}_3 = 0, \; \tilde{x}_1 \in \mathbb{R}$$
$$\Rightarrow \vec{x} = P\tilde{x} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} \tilde{x}_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \tilde{x}_1 \\ 0 \\ \tilde{x}_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\tilde{x}_1 \in \mathbb{R}$$

3) $$A = PDP^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix} =$$
$$= \begin{bmatrix} -1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & -1/2 \end{bmatrix} \Rightarrow [A \mid \vec{0}] = \left[\begin{array}{ccc|c} -1/2 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 0 \\ 1/2 & 0 & -1/2 & 0 \end{array}\right] \sim$$
$$\sim \left[\begin{array}{ccc|c} -1/2 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_2 = 0 \\ x_1 = x_3 \\ x_3 \in \mathbb{R} \end{array} \Rightarrow \vec{x} = \begin{bmatrix} x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, x_3 \in \mathbb{R}$$

4)
~~$A\vec{x} = \vec{0} \Rightarrow \vec{x}$~~
$$A\vec{x} = \lambda\vec{x} = 0 \cdot \vec{x}$$
$\Rightarrow x$ is $\vec{e}_1$ up to a constant
$$\Rightarrow \vec{x} = c\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, c \in \mathbb{R}$$