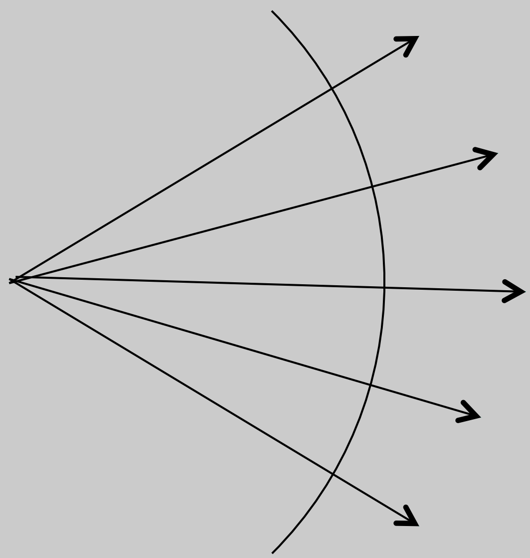
Curved wavefronts

- Rays are directed normal to surfaces of constant phase
  - These surfaces are the wavefronts
  - Radius of curvature is approximately at the focal point



- Spherical waves are solutions to the wave equation (away from r = 0)

$$\nabla^2 E + \frac{n^2\omega^2}{c^2} E = 0$$

$$E \propto \frac{1}{r} e^{i(\pm kr - \omega t)}$$

$$I \propto \frac{1}{r^2}$$

Scalar r
+ outward
- inward

Paraxial approximations

- For **rays**, paraxial = small angle to optical axis
  - Ray slope: $\tan\theta \approx \theta$
- For **spherical waves** where power is directed forward:

$$e^{ikr} = \exp\left[ik\sqrt{x^2+y^2+z^2}\right]$$

$$k\sqrt{x^2+y^2+z^2} = kz\sqrt{1+\frac{x^2+y^2}{z^2}} \approx kz\left(1+\frac{x^2+y^2}{2z^2}\right)$$

Expanding to 1st order

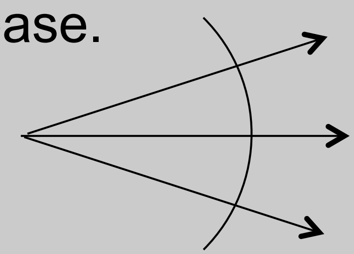
$$e^{i(kr-\omega t)} \to e^{ikz}\exp\left[i\left(k\frac{x^2+y^2}{2z}-\omega t\right)\right]$$

z is radius of curvature

Wavefront = surface of constant phase $k\frac{x^2+y^2}{2z} = \omega t$

For x, y >0, t must increase.

Wave is diverging:



3D wave propagation

$$\nabla^2 \mathbf{E} - \frac{n_j^{\ 2}}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E} = \frac{\partial^2}{\partial z^2}\mathbf{E} + \nabla_\perp^{\ 2}\mathbf{E} - \frac{n(\mathbf{r})^2}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E} = 0$$

$$\nabla_\perp^2 = \partial_x^2 + \partial_y^2$$

$$\boxed{\nabla_\perp^2 = \frac{1}{r}\partial_r\left(r\,\partial_r\right) + \frac{1}{r^2}\partial_\phi^2}$$

- Note:
  - All linear propagation effects are included in LHS: diffraction, interference, focusing…
  - Previously, we assumed plane waves where transverse derivatives are zero.
- More general examples:
  - Gaussian beams (including high-order)
  - Waveguides
  - Arbitrary propagation
  - Can determine discrete solutions to linear equation (e.g. Gaussian modes, waveguide modes), then express fields in terms of those solutions.

Diffractive propagation

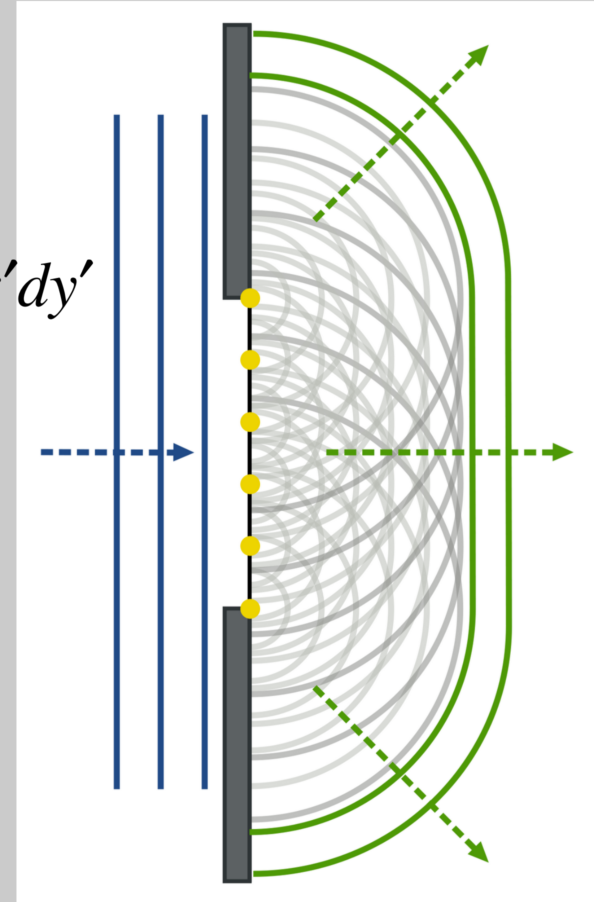
- Huygens' principle:
  - Represent a plane wave as a superposition of source points emitting spherical waves
- Integral representation:

$$E(x,y,z)=\frac{i}{\lambda}\iint \underbrace{E(x',y',z')}_{\text{Field at input plane}}\underbrace{\frac{\exp\left[-ik\left|\mathbf{r}-\mathbf{r}'\right|\right]}{\left|\mathbf{r}-\mathbf{r}'\right|}}_{\text{Spherical wavelet}}\underbrace{\cos\theta}_{\text{Inclination factor}}\,dx'dy'$$

Field at input plane

Spherical wavelet

Inclination factor



This is essentially a convolution of the complex input field with the spherical wavelets, which are the Green's function for the wave equation

Paraxial, slowly-varying approximations

- Assume
  - waves are forward-propagating:
    $\mathbf{E}(\mathbf{r},t) = \mathbf{A}(\mathbf{r})e^{i(kz-\omega_0 t)} + \text{c.c.}$
  - Refractive index is isotropic

$$\frac{\partial^2}{\partial z^2}\mathbf{A} + 2ik\frac{\partial}{\partial z}\mathbf{A} - k^2\mathbf{A} + \nabla_\perp{}^2\mathbf{A} + \frac{n^2\omega_0{}^2}{c^2}\mathbf{A} = 0$$

  - Fast oscillating carrier terms cancel (blue)
- Slowly-varying envelope: compare red terms
  - Drop 2nd order deriv if $\frac{2\pi}{\lambda}\frac{1}{L}A \gg \frac{1}{L^2}A$
  - This ignores:
    - Changes in z as fast as the wavlength
    - Counterpropagating waves

$$2ik\frac{\partial}{\partial z}\mathbf{A} + \nabla_\perp{}^2\mathbf{A} = 0$$

Fresnel diffraction integral

- Fresnel approximation (near field)
  - Expand the spherical wave in paraxial approximation (in exponential)
  - Let denominator be $\left|\mathbf{r}-\mathbf{r}'\right| \sim z-z'=L \quad \cos\theta \simeq 1$
  - Input field: $E\left(x',y',z'\right)=u\left(x',y',z'\right)e^{-ik\left(z-z'\right)}$

$$u\left(x,y,z\right)=\frac{i}{\lambda L}\iint u\left(x',y',z'\right)\exp\left[-ik\frac{\left(x-x'\right)^2+\left(y-y'\right)^2}{2L}\right]dx'dy'$$

$$u\left(x,y,z\right)=\frac{i}{\lambda L}\mathrm{e}^{-ik\frac{x^2+y^2}{2L}}\iint u\left(x',y',z'\right)\mathrm{e}^{-ik\frac{x'^2+y'^2}{2L}}\,\mathrm{e}^{-i\frac{k}{L}\left(xx'+yy'\right)}\,dx'dy'$$

Fraunhofer diffraction integral

$$u(x,y,z)=\frac{i}{\lambda L}\mathrm{e}^{-ik\frac{x^2+y^2}{2L}}\iint u(x',y',z')\mathrm{e}^{-ik\frac{x'^2+y'^2}{2L}}\ \mathrm{e}^{-i\frac{k}{L}(xx'+yy')}dx'dy'$$

- In the "far field", we approximate the sum of paraxial spherical waves as a sum of plane waves
  - Assume field in input plane is confined to a radius *a*
  - If $\frac{ka^2}{2L}=\frac{\pi a^2}{\lambda}\frac{1}{L}<<1$ then we drop quadratic phases.

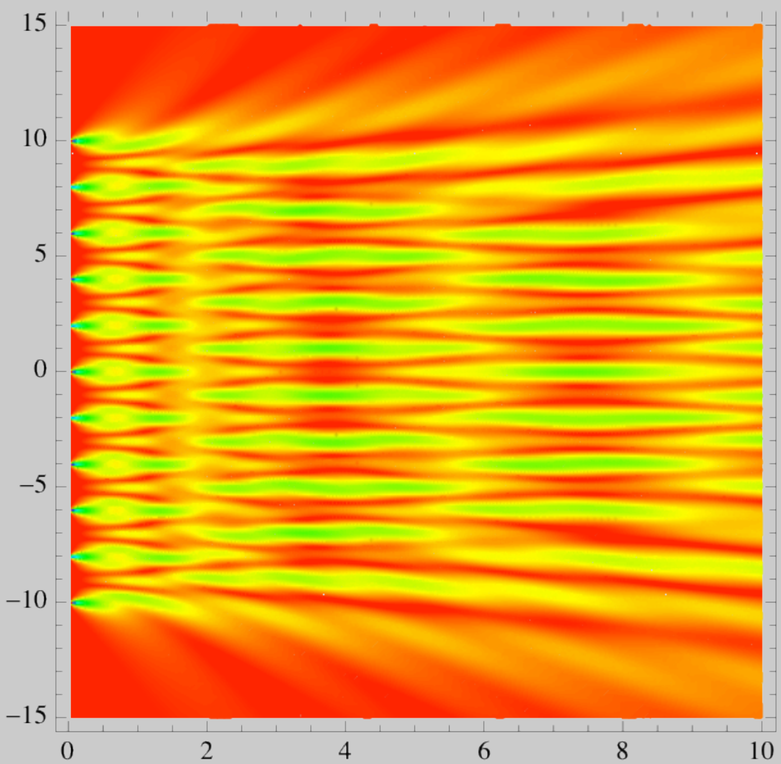
$$u(x,y,z)=\frac{i}{\lambda L}\iint u(x',y',z')\exp\left[-i\left(\frac{kx}{L}x'+\frac{ky}{L}y'\right)\right]dx'dy'$$

  - Result: far field is a Fourier transform of the input field
  - "spatial frequencies" $\beta_x=k\frac{x}{L}=k\sin\theta_x \quad \beta_y=k\frac{y}{L}=k\sin\theta_y$

Example: sum of dipole radiators

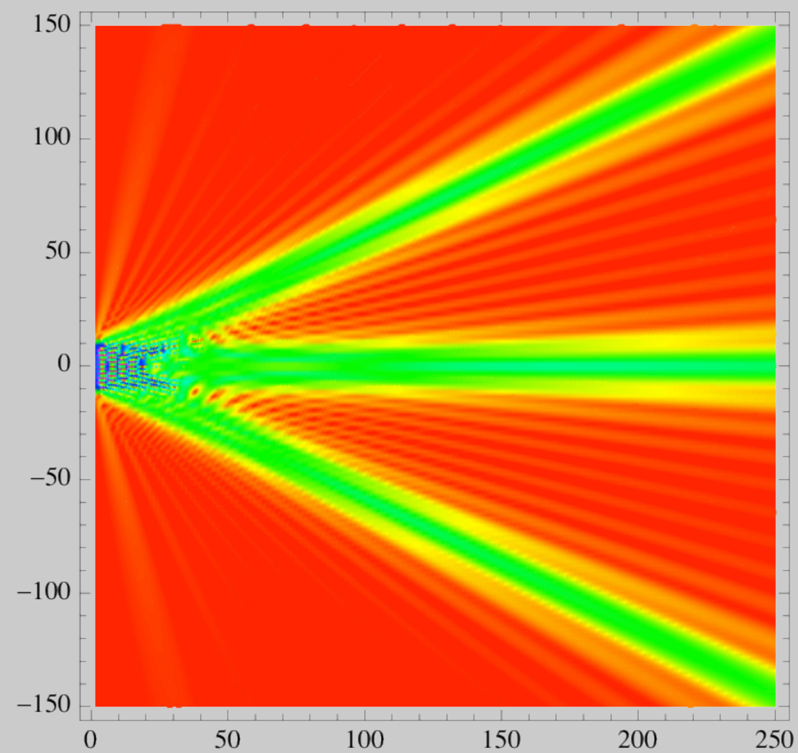
- Add fields from 10 individual sources

Near field



Talbot fringes

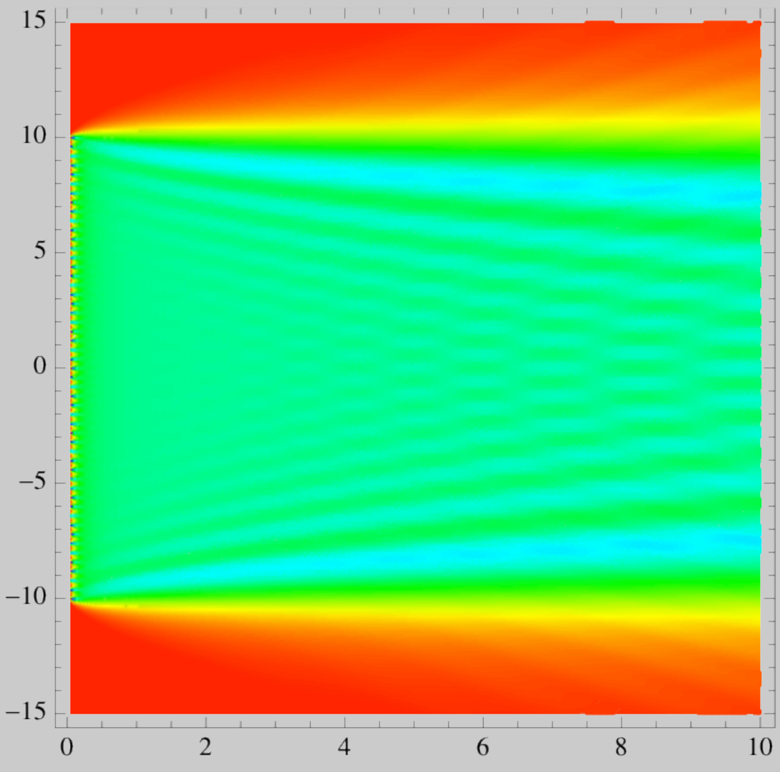
far field

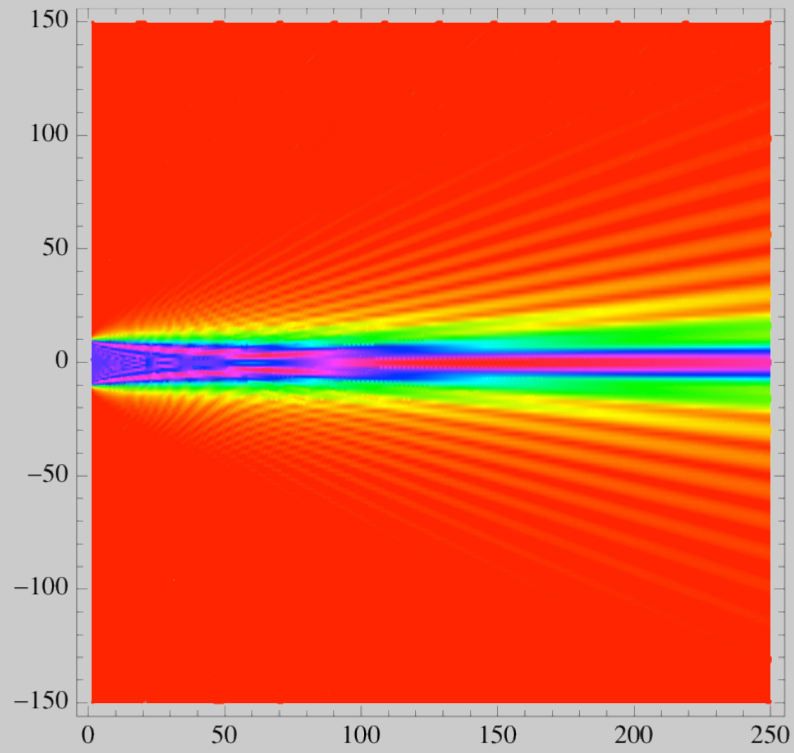


Diffraction grating

High-density of radiators

- Combine 50 sources over same distance



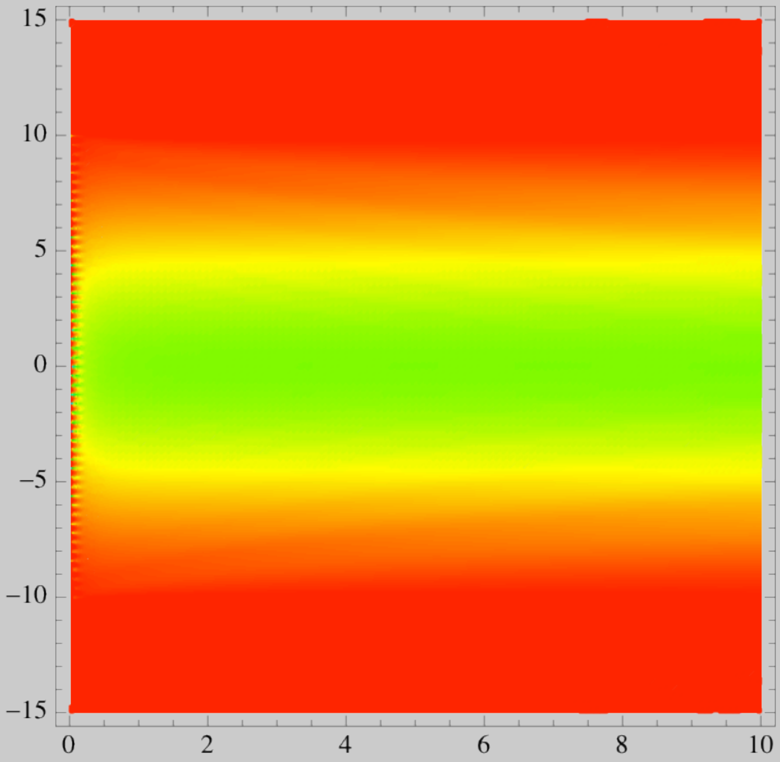


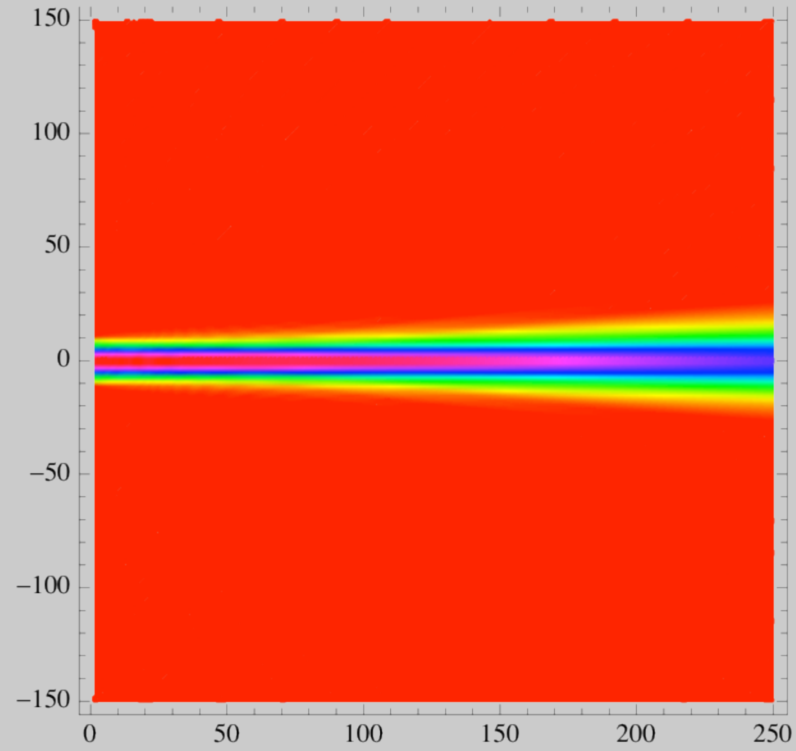
Fresnel zone shows shadow boundary, diffraction fringes

Far field evolves more like a beam, with single-slit diffraction.

High density of radiators, Gaussian envelope

- Gaussian amplitude envelope eliminates diffraction fringes





Beam smoothly spreads out with distance

Gaussian beam solution to wave equation

- Use Fresnel integral to propagate a Gaussian beam

$$u(x,y,z)=\frac{i}{\lambda L}\mathrm{e}^{-ik\frac{x^2+y^2}{2L}}\iint e^{-\frac{x'^2+y'^2}{w^2}}\,\mathrm{e}^{-ik\frac{x'^2+y'^2}{2L}}\,\mathrm{e}^{-i\left(\beta_x x'+\beta_y y'\right)}\,dx'dy'$$

  - Combine quadratic terms in exponent:

$$\left(\frac{1}{w^2}+i\frac{k}{2L}\right)=i\frac{k}{2}\left(\frac{1}{L}-i\frac{2}{kw^2}\right)=i\frac{k}{2q}$$

  - Now integral is a F.T. of a complex Gaussian=Gaussian

$$u(x,y,z)=\frac{i}{\lambda L}\mathrm{e}^{-ik\frac{x^2+y^2}{2L}}\iint \mathrm{e}^{-ik\frac{x'^2+y'^2}{2q}}\,\mathrm{e}^{-i\left(\beta_x x'+\beta_y y'\right)}\,dx'\,dy'$$

Standard form of Gaussian beam equations

$$E(r,z,t) = A_0 e^{-i(kz-\omega t)} \frac{w_0}{w(z)} e^{-\frac{r^2}{w^2(z)}} e^{-i\frac{kr^2}{2R(z)}} e^{i\eta(z)}$$

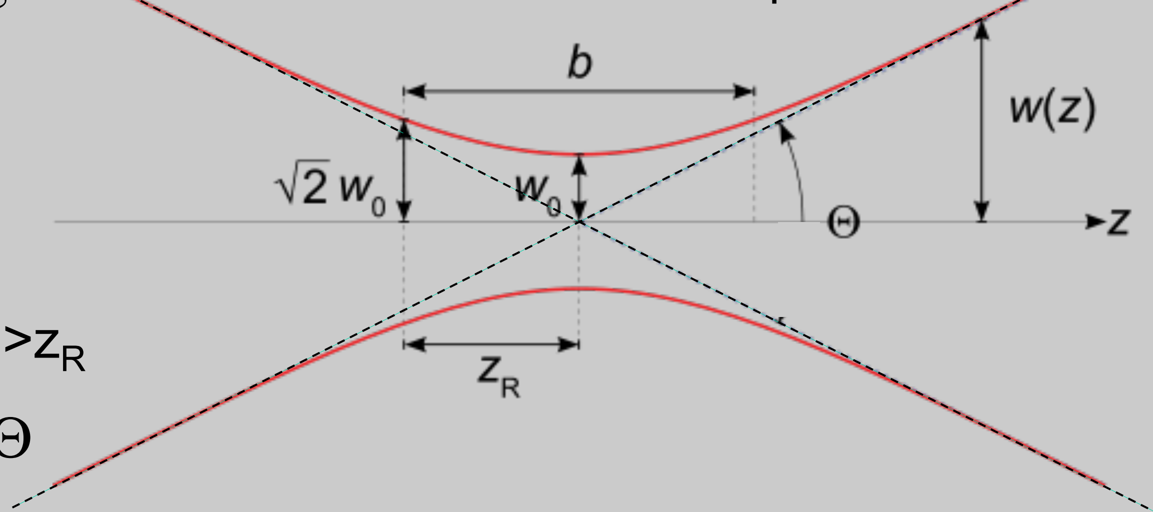
Beam maintains a Gaussian profile as it propagates

- beam radius that varies with z
- Origin of z coordinate is at the beam waist
- Rayleigh length z_R defines collimation distance from focal plane

$$z_R = \frac{\pi w_0^2}{\lambda / n} = \frac{\text{beam area}}{\text{wavelength in medium}}$$

b = confocal parameter
= depth of focus

$$w(z) = w_0 \sqrt{1 + \frac{z^2}{z_R^{\ 2}}}$$



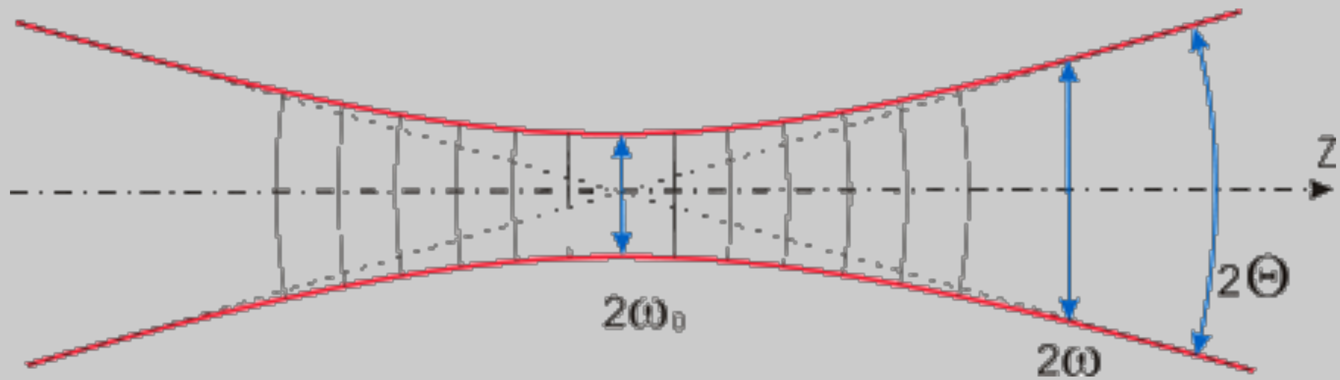
Geometric limit for $z \gg z_R$

$$w(z) = z \frac{w_0}{z_R} = z \tan\Theta$$

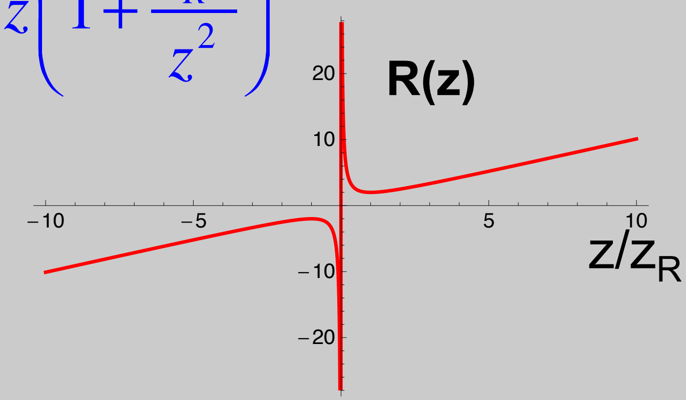
Evolution of wavefronts

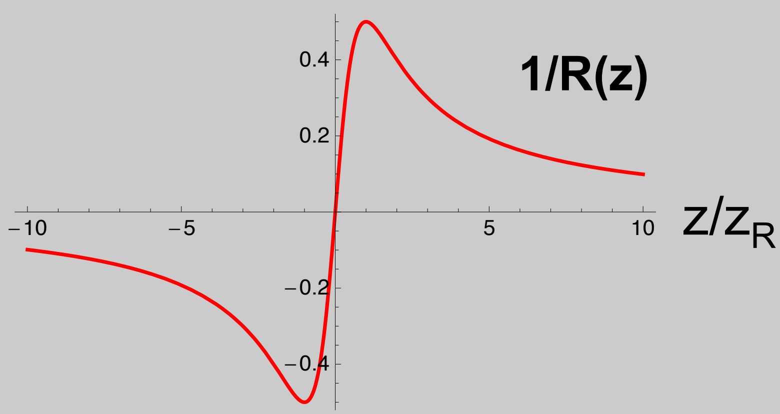
$$E(r,z,t) = A_0 e^{-i(kz-\omega t)} \frac{w_0}{w(z)} e^{-\frac{r^2}{w^2(z)}} e^{-i\frac{kr^2}{2R(z)}} e^{i\eta(z)}$$

- Wavefront curvature evolves with z as beam size changes



$$R(z) = z\left(1 + \frac{z_R^{\,2}}{z^2}\right)$$





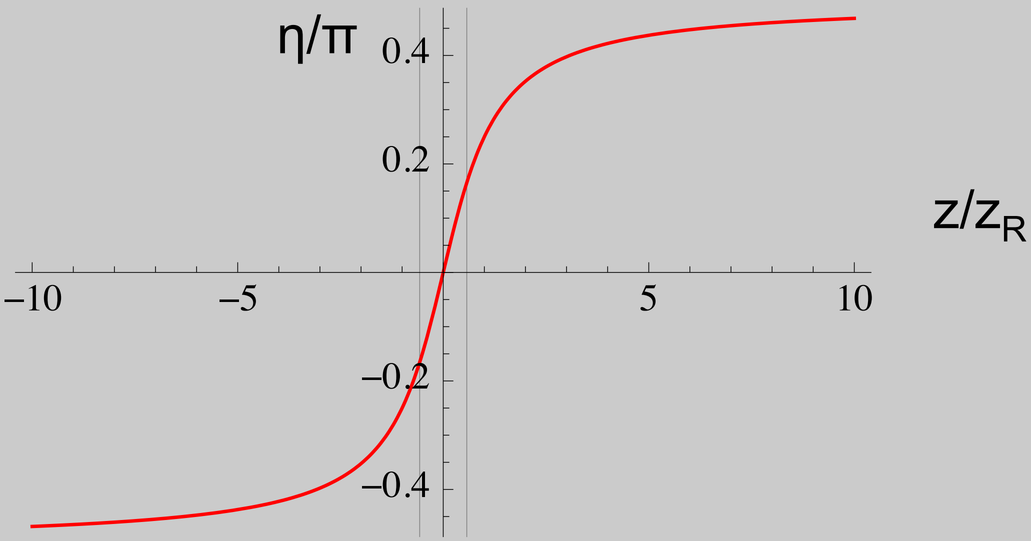
On-axis phase: Gouy phase

$$E(r,z,t) = A_0 e^{-i(kz-\eta(z)-\omega t)} \frac{w_0}{w(z)} e^{-\frac{r^2}{w^2(z)}} e^{-i\frac{kr^2}{2R(z)}}$$

- Because the wavefront changes from focusing to defocusing, on-axis phase advances with z

Gouy phase

$$\eta(z) = \arctan\left(\frac{z}{z_R}\right)$$



Higher-order Hermite-Gauss modes

– The Gaussian beam is just the lowest order mode solution to the wave equation

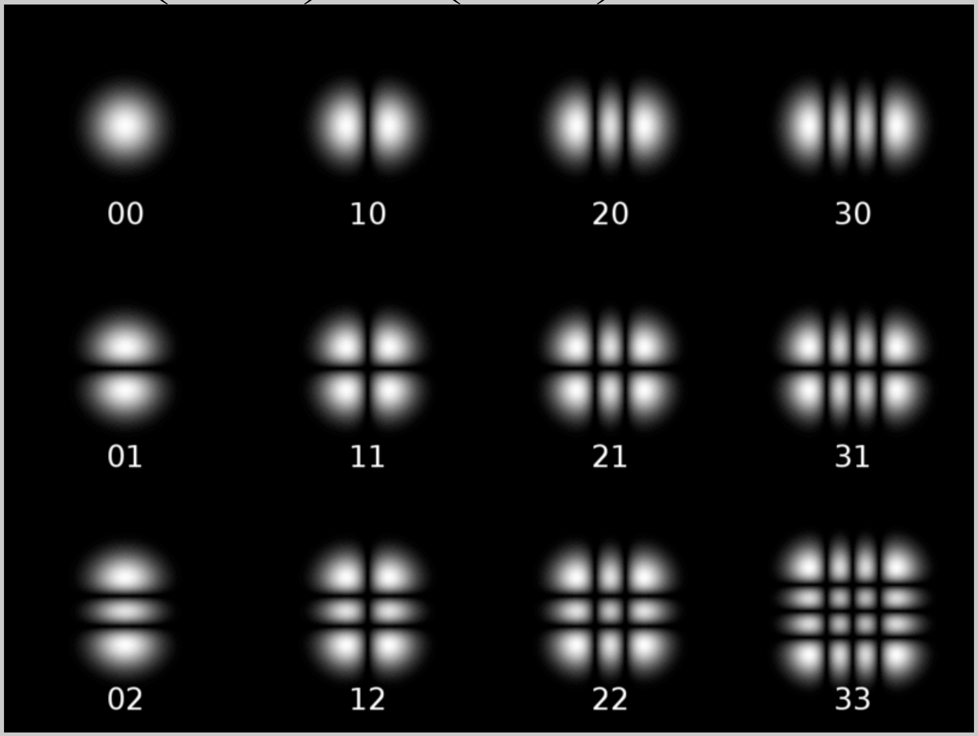
– x, y coordinates: Hermite-Gaussian modes

$$E(x,y,z) = A_0 e^{-i(kz-\eta_{lm}(z))} \frac{w_0}{w(z)} e^{-\frac{x^2+y^2}{w^2(z)}} H_l\left(\frac{\sqrt{2}x}{w(z)}\right) H_m\left(\frac{\sqrt{2}y}{w(z)}\right) e^{-i\frac{k(x^2+y^2)}{2R(z)}}$$

$$\eta_{lm} = (l+m+1)\tan^{-1}\left(\frac{z}{z_R}\right)$$

Transverse profile is maintained during propagation (scaled with w(z))

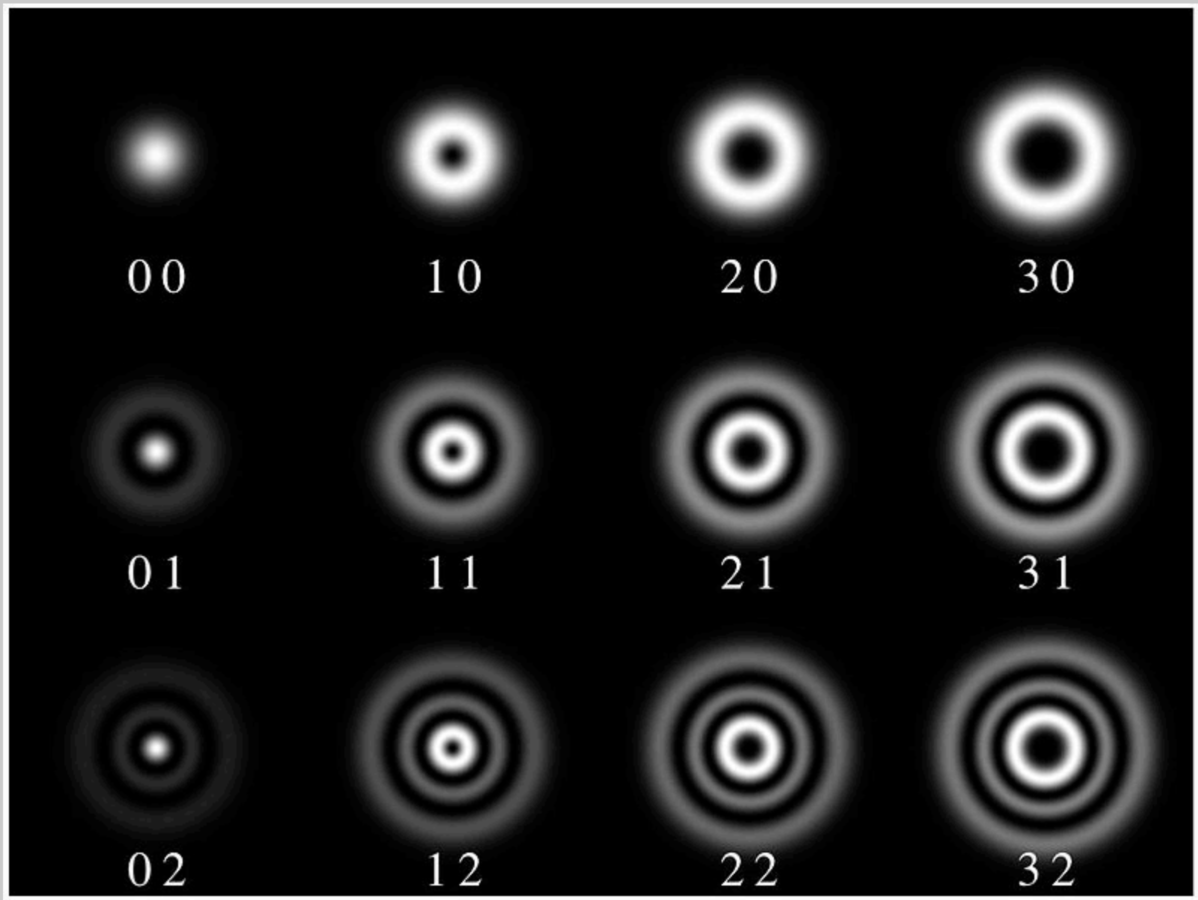
Hermite-Gauss functions are the same as solutions to quantum SHO



Higher-order LaGuerre-Gauss modes

– In cylindrical coordinates, alternate representation
– Azimuthal phase $\exp[im\phi]$ "vortex" phase

Example:
LG10 mode is a linear combination of HG10 and HG01



Complex q vs standard form

$$u(r,z) = \frac{z_R}{q(z)} e^{-ik\frac{r^2}{2q(z)}}$$ with $$\frac{1}{q(z)} = \frac{1}{R(z)} - i\frac{\lambda}{\pi w^2(z)}$$

Expand exponential:

$$\exp\left[-ik\frac{r^2}{2q(z)}\right] = \exp\left[-ik\frac{r^2}{2}\left(\frac{1}{R(z)} - i\frac{\lambda}{\pi w^2(z)}\right)\right]$$

$$= \exp\left[-ik\frac{r^2}{2}\frac{1}{R(z)} - i\frac{2\pi}{\lambda}\frac{r^2}{2}\left(-i\frac{\lambda}{\pi w^2(z)}\right)\right] = e^{-ik\frac{r^2}{2R(z)}} e^{-\frac{r^2}{w^2(z)}}$$

$$a + ib = \sqrt{a^2 + b^2}\, e^{i\arctan(b/a)}$$

Expand leading inverse q:

$$\frac{1}{q(z)} = \left(\frac{z}{z^2 + z_R{}^2} - i\frac{z_R}{z^2 + z_R{}^2}\right) = -i\left(\frac{z_R + iz}{z^2 + z_R{}^2}\right) = -i\left(\frac{\sqrt{z^2 + z_R{}^2}}{z^2 + z_R{}^2}\right) e^{i\arctan(z/z_R)}$$

$$= -i\left(\frac{1}{z_R\sqrt{1 + z^2 / z_R{}^2}}\right) e^{i\arctan(z/z_R)} = \frac{w_0}{i z_R w(z)} e^{i\eta(z)}$$

Gaussian beams and ABCD

- General expression

$$q_1 = \frac{Aq_0 + B}{Cq_0 + D} \qquad \frac{1}{q(z)} = \frac{1}{R(z)} - i\frac{\lambda}{\pi w^2(z)}$$

  - Since q is defined through its inverse, alternate:

$$q_1^{-1} = \frac{C + Dq_0^{-1}}{A + Bq_0^{-1}}$$

  - Note that ABCD matrices are the same as for raytrace
  - Application is **not** a multiplication like matrix.vector

Simple examples

- translation

$$\frac{1}{q(z)} = \frac{1}{R(z)} - i\frac{\lambda}{\pi w^2(z)} = \frac{1}{z\left(1+\frac{z_R{}^2}{z^2}\right)} - i\frac{\lambda}{\pi w_0{}^2\left(1+\frac{z^2}{z_R{}^2}\right)}$$

$$R(z) = z\left(1+\frac{z_R{}^2}{z^2}\right)$$

$$w(z) = w_0\sqrt{1+\frac{z^2}{z_R{}^2}}$$

$$= \frac{1}{z_R\left(1+\frac{z^2}{z_R{}^2}\right)}\left(z/z_R - i\right)$$

$$q(z) = z_R\left(1+\frac{z^2}{z_R{}^2}\right)\frac{1}{\left(z/z_R - i\right)}$$

$$= z_R\left(1+\frac{z^2}{z_R{}^2}\right)\frac{z/z_R + i}{\left(1+\frac{z^2}{z_R{}^2}\right)} = z + iz_R$$

$$q(z) = z + iz_R \rightarrow q_1 = z_0 + L + iz_R$$

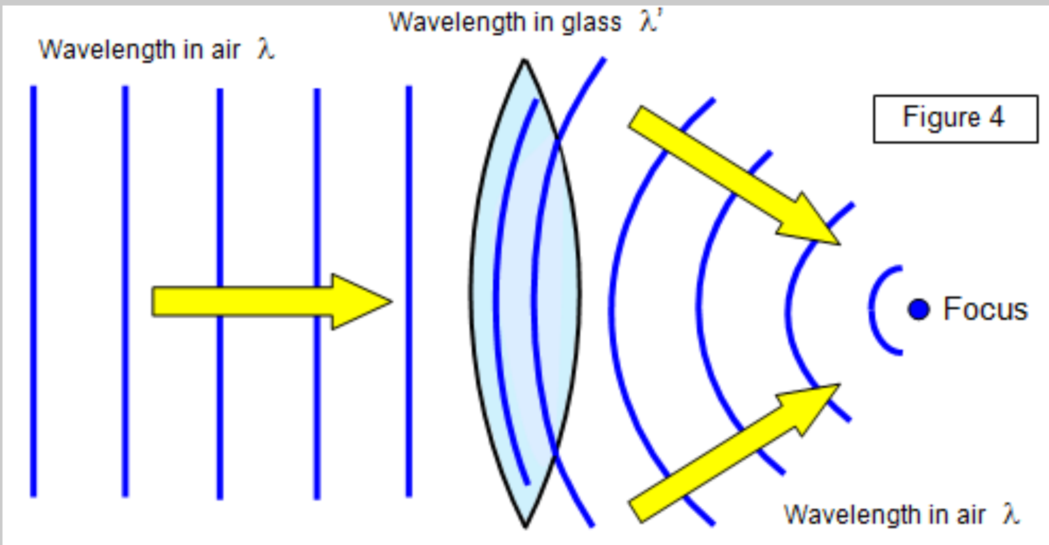
ABCD for translation:

$$\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

$$q_1 = \frac{Aq_0 + B}{Cq_0 + D} = q_0 + L$$

Simple examples

- Focusing by a lens
  - Radius of curvature is modified by lens: $\frac{1}{R'} = \frac{1}{R} - \frac{1}{f}$



Focusing by lens induces a negative ROC

$$\frac{1}{q_1} = \frac{1}{R} - \frac{1}{f} - i\frac{\lambda}{\pi w^2} = \frac{1}{q_0} - \frac{1}{f}$$

$$q_1^{-1} = \frac{C + Dq_0^{-1}}{A + Bq_0^{-1}}$$

$$\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$$

Focusing a Gaussian beam by a lens

- For a beam waist at lens entrance, distance from lens to focused waist is not exactly = f
- Define variables:

  w_{01} (w_{02}) = input (focused) beam waist radius

  z_{R1} (z_{R2}) = rayleigh range for input (focused) beam

  z_m = distance from lens to focused beam waist
- Use Gaussian beam equations to back propagate to lens

$$w_{01} = w\left(z = -z_m\right) = w_{02}\sqrt{1 + \frac{z_m^{\ 2}}{z_{R2}^{\ 2}}} \qquad \rightarrow z_{R1} = \frac{\pi w_{01}^{\ 2}}{\lambda} = z_{R2}\left(1 + \frac{z_m^{\ 2}}{z_{R2}^{\ 2}}\right)$$

$$R\left(z = -z_m\right) = -f = -z_m\left(1 + \frac{z_{R2}^{\ 2}}{z_m^{\ 2}}\right)$$

- Divide equations:

$$\rightarrow z_{R1} = \frac{\pi w_{01}^{\ 2}}{\lambda} = z_{R2}\left(1 + \frac{z_m^{\ 2}}{z_{R2}^{\ 2}}\right)$$