

Diffraction

Intuitive concept: Huygen's point sources across aperture
 - needs refinement

scalar spherical wave: $E(r) = E_0 \frac{e^{ikr}}{r}$

- E_0 = complex (e.g. $A e^{i\phi}$)
- take real part for real field.
- suppressing time dependence.
- note intensity $\sim E^2 \sim 1/r^2$

Implement intuitive idea: even all Gaussians, diverge aperture.

let $\Psi_{in}(\vec{r})$ be scalar field in aperture plane ($x, y, 0$)
 - simplest form $\rightarrow \text{rect}(.)$

diffracted field is the convolution of spherical wave with
 input aperture wave:

$$\Psi_{diff}(\vec{R}) \propto \left\{ \Psi_{in}(\vec{r}) \frac{e^{ik|\vec{R}-\vec{r}|}}{|\vec{R}-\vec{r}|} \right\} dS$$



- fixed point \vec{R} results from sum across aperture
- note phase changes are captured $e^{i k |\vec{R}-\vec{r}|}$

here $\Psi_{in}(\vec{r}) = \text{rect}(x/a) \text{rect}(y/a)$ as example.

Is this legitimate? Yes, but...

Gaussian analysis $\rightarrow \Psi_{diff}(\vec{R}) = \frac{1}{ik} \int dS \Psi_{in}(\vec{r}) e^{ik|\vec{R}-\vec{r}|} \frac{(1 + \cos \theta)}{\vec{R} \cdot \vec{r}}$

$\frac{1}{2}$ = phase shift $\frac{1}{2}$ for units

$\frac{1}{2}(1 + \cos \theta)$ = "obliquity factor" ≈ 1 to make sure light \rightarrow forward.

Fresnel + Fraunhofer diffraction

- propagation of the scalar field is calculated by doing a convolution of the input field $\Psi_{\text{inc}}(\vec{r})$ with the virtual source points $\frac{1}{2}(1 + \cos\theta) \frac{e^{ik|\vec{R}-\vec{r}|}}{|\vec{R}-\vec{r}|}$
- work with small angles so that $\frac{1}{2}(1 + \cos\theta) \approx 1$
 $(\frac{1}{4}\theta^2 \ll 1 \quad \theta \approx \Delta x/R)$
- coordinates: starting plane $Z=0, X, Y$
end plane Z, X, Y
- typically, there is an aperture at $Z=0$
 ↗ but there does not have to be. Just require that:
 - light propagates mostly in forward direction
 - beam is spatially bounded

Integral is same as we saw in antenna thy:

$$|\vec{R}-\vec{r}| = \sqrt{(X-x)^2 + (Y-y)^2 + Z^2}$$

$$\text{factor out } R = \sqrt{X^2 + Y^2 + Z^2}$$

$$\begin{aligned} |\vec{R}-\vec{r}| &= R \left[\frac{X^2 + Y^2 + Z^2 - 2Xx - 2Yy + x^2 + y^2}{R^2} \right]^{1/2} \\ &= R \left[1 + \frac{x^2 + y^2}{R^2} - 2 \frac{(Xx + Yy)}{R^2} \right]^{1/2} \end{aligned}$$

in input plane, $X, Y < a$ aperture or beam radius
 For $a \ll R$

$$\text{and } (Xx + Yy)/R^2 \ll 1$$

approx $\sqrt{1 - 2(Xx + Yy)/R^2}$

in phase term:

$$k|\vec{R}-\vec{r}| \approx kR \left(1 + \frac{x^2+y^2}{2R^2} - \frac{Rx+Ry}{R^2}\right)$$

let $\gamma/|\vec{R}-\vec{r}| \rightarrow \gamma/R$ field transmitted past starting plane.

$$\Psi_{\text{dif}}(\vec{x}, \vec{z}) = \frac{1}{i\lambda} \iint \Psi_{\text{tr}}(x, y) \frac{e^{ikR}}{R} e^{ik\frac{R}{2R}(x^2+y^2)} e^{-ik\frac{(Rx+Ry)}{R}} dx dy$$

this is the Fresnel integral ("near field")

if $\frac{k\alpha^2}{2R} = \frac{\pi\alpha^2}{\lambda} \cdot \frac{1}{R} \ll 1$, we can drop terms w/ quadratic phase.

→ Fraunhofer diffraction. ("far field")

Fresnel: convolution w/ paraxial appx to spherical waves

Fraunhofer: sum up plane waves from source plane to output.

if $y = \vec{z} = 0$ $-i\frac{k\vec{z}}{R}$ is like a plane wave

$$\vec{z}/R \approx \sin \theta_x$$

$$\frac{k\vec{z}}{R} \approx k \sin \theta_x = k_x \equiv \beta_x \quad \text{this is called a spatial frequency variable.}$$

with this change we can write the Fraunhofer integral as:

$$\Psi_{\text{dif}}(\vec{x}, \vec{z}) = \frac{e^{ikR}}{i\lambda R} \iint \Psi_{\text{tr}}(x, y) e^{-i(\beta_x x + \beta_y y)} dx dy$$

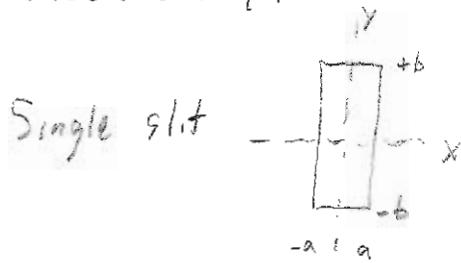
the integral is a 2-D Fourier transform.

note that the derivation has given us a different sign!

$$\text{define } F(\beta_x, \beta_y) = \mathcal{F}_{xy} \{ f(x, y) \} = \iint f(x, y) e^{-i\beta_x x} e^{-i\beta_y y} dx dy$$

$$\text{inverse } f(x, y) = \mathcal{F}_{xy}^{-1} \{ F(\beta_x, \beta_y) \} = \frac{1}{4\pi^2} \iint F(\beta_x, \beta_y) e^{+i\beta_x x} e^{+i\beta_y y} d\beta_x d\beta_y$$

Using Fourier transforms makes the calculation of diffraction much easier.



plane wave incident at \$z=0\$
transmitted field:

$$E_{tr}(x, y) = E_{inc}(x, y) A(x, y)$$

$$= E_0 \text{rect}\left(\frac{x}{2a}\right) \text{rect}\left(\frac{y}{2b}\right)$$

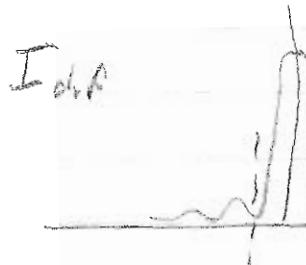
direct integral:

$$\begin{aligned} E_{dir}(\bar{x}, \bar{y}) &= \frac{e^{i\bar{k}R}}{i\lambda R} \iint_{-\infty}^{\infty} E_0 A(x, y) e^{-i\frac{\bar{k}}{R}(\bar{x}x + \bar{y}y)} dx dy \\ &= \frac{E_0 e^{i\bar{k}R}}{i\lambda R} \int_a^a dx e^{-i\bar{k}\bar{x}x/R} \int_{-b}^b dy e^{-i\bar{k}\bar{y}y/R} \\ &= \frac{E_0 e^{i\bar{k}R}}{i\lambda R} \left(\frac{e^{-i\bar{k}\bar{x}a/R} - e^{+i\bar{k}\bar{x}a/R}}{-i\bar{k}\bar{x}/R} \right) \left(\text{similar for } y \right) \\ &= \frac{E_0 e^{i\bar{k}R}}{i\lambda R} \left(\frac{2 \sin(k\bar{x}a/R)}{k\bar{x}/R} \right) \left(\frac{2 \sin(k\bar{y}b/R)}{k\bar{y}/R} \right) \\ &= E_0 e^{i\bar{k}R} \frac{4ab}{i\lambda R} \text{sinc}\left(\frac{k\bar{x}a}{R}\right) \text{sinc}\left(\frac{k\bar{y}b}{R}\right) \end{aligned}$$

by F.T.:

$$\begin{aligned} E_{dir}(\bar{x}, \bar{y}) &= \frac{E_0 e^{i\bar{k}R}}{i\lambda R} \underbrace{\{ \text{rect}(x/2a) \}}_{x} \underbrace{\{ \text{rect}(y/2b) \}}_{y} \\ &= \frac{E_0 e^{i\bar{k}R}}{i\lambda R} 4ab \text{sinc}(\beta_x a) \text{sinc}(\beta_y b) \end{aligned}$$

$$\text{intensity} \propto |E_{dir}(\bar{x}, \bar{y})|^2 \sim E_0^2 \left(\frac{4ab}{\lambda R} \right)^2 \text{sinc}^2(\beta_x a) \text{sinc}^2(\beta_y b)$$



1st zero at

$$k_x a = \pi \rightarrow \frac{k_x a}{R} = \pi$$

$$\frac{X}{R} = \frac{\lambda}{2a}$$

pattern spreads out as R^2



$$\frac{X}{R} = \sin \theta_x = \frac{\lambda}{2a} \quad \text{diffraction angle.}$$

Scaling: small aperture, wider spread angle.

higher index medium? $k \rightarrow \frac{nw}{c}$ $\sin \theta_x = \frac{\lambda}{2a} \cdot \left(\frac{1}{n}\right)$
 \rightarrow smaller θ_x

Uncertainty principle

$$\text{in QM } \Delta x \Delta p_x \geq \hbar/2$$

here $\Delta x = 2a$ the slit

$$\Delta p_x = \hbar \Delta k_x = \hbar k_x \sin \theta_x = \hbar \frac{3\pi}{\lambda} \cdot \frac{\lambda}{2a} = \frac{\hbar 3\pi}{2a}$$

$$\Delta x \Delta p_x = \hbar \quad \text{or} \quad \Delta x \Delta k_x = 2\pi$$

Double slit. $E_{in}(x,y) = E_0$. $A = \text{rect}\left(\frac{x-d}{2a}\right) + \text{rect}\left(\frac{x+d}{2a}\right)$

drop products one arm.

$$E_{out} \propto \mathcal{F} \left\{ \text{rect}\left(\frac{x-d}{2a}\right) + \text{rect}\left(\frac{x+d}{2a}\right) \right\}$$

just like double pulse

$$E_{out} \propto a e^{i k d} \sin(\beta x^2) + e^{-i k d} \sin(\beta x^2)$$

$$= 2 \underbrace{\sin(\beta x^2)}_{\text{broad envelope}} \underbrace{\cos(kd)}_{\text{interference fringes}}$$

Q $A(x) = \text{rect}\left(\frac{x}{2a}\right) \otimes (\delta(x-d) + \delta(x+d))$

$$\mathcal{F}\{A(x)\} = \mathcal{F}\{\text{rect}\left(\frac{x}{2a}\right)\} \mathcal{F}\{\delta(x-d) + \delta(x+d)\}$$

Non-normal incidence

same $A(x)$



$$E_{in} \sim e^{i k z}$$

$$i(kz + k_x z)$$

$$k_z = k \cos \theta$$

$$k_x = k \sin \theta$$

evaluate at $z=0$

$$E_{in} \sim e^{i k z \sin \theta}$$

Now output has a shift

$$E_{out} \sim \mathcal{F} \{ e^{i k z \sin \theta} A(x) \}$$

$$\beta_x \rightarrow \beta_x + k z \sin \theta$$

$$\text{since } \beta_x = -k \sin \theta \quad \beta'_x = \beta_x (\sin \theta + k z \sin \theta)$$

$$= k z (\sin \theta + \beta_x)$$

max at $\beta' = 0$ or $z = R \sin \theta$