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Fourier transforms  
pulse propagation

# Fourier transforms: t- $\omega$ domain

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{+i\omega t} dt = FT\{f(t)\}$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{-i\omega t} d\omega = FT^{-1}\{F(\omega)\}$$

- In EM, our signals are complex fields
- $1/2\pi$  factor is lumped into inverse transform
- $\omega$  is our frequency variable, not v. This affects the normalization constants
- Note signs of exponents: this is tied to our  $\exp(-i \omega t)$  convention
- Techniques
  - Analytic: apply transform IDs and theorems to decompose a transform into its parts
  - Analytic in Mathematica: can do some FTs but not always expressed in recognizable way
  - Graphical: after identifying components of a transform, sketch the anticipated result
  - Numerical: FFT for calculating complicated or realistic cases for modeling/data analysis

# FT of a Gaussian

- Starting integral:  $\int e^{-z^2} dz = \sqrt{\pi}$ 
  - True even if  $z$  is complex

$$f(t) = e^{-t^2/t_0^2} \quad FT\{f(t)\} = F(\omega) = \int_{-\infty}^{\infty} e^{-t^2/t_0^2} e^{+i\omega t} dt$$

- Complete the square in the exponent...

# FT of a Gaussian is a Gaussian

- Starting integral:  $\int e^{-z^2} dz = \sqrt{\pi}$ 
  - True even if  $z$  is complex

$$f(t) = e^{-t^2/t_0^2} \quad FT\{f(t)\} = F(\omega) = \int_{-\infty}^{\infty} e^{-t^2/t_0^2} e^{+i\omega t} dt$$

- Complete the square in the exponent

$$-\frac{t^2}{t_0^2} + i\omega t = -\frac{1}{t_0^2} \left( t^2 - i\omega t t_0^2 \right) = -\frac{1}{t_0^2} \left( \left( t - \frac{i}{2} \omega t_0^2 \right)^2 + \frac{1}{4} \omega^2 t_0^4 \right)$$

$$= -\frac{1}{t_0^2} \left( t - \frac{i}{2} \omega t_0^2 \right)^2 - \frac{1}{4} \omega^2 t_0^2$$

- Change variables:  $z = \frac{1}{t_0} \left( t - \frac{i}{2} \omega t_0^2 \right)$

$$F(\omega) = \int_{-\infty}^{\infty} e^{-t^2/t_0^2} e^{+i\omega t} dt = t_0 e^{-\frac{1}{4} \omega^2 t_0^2} \int_{-\infty}^{\infty} e^{-z^2} dz = \sqrt{\pi} t_0 e^{-\frac{1}{4} \omega^2 t_0^2}$$

# Time-bandwidth product

- “uncertainty principle” comes from FT relations

$$FT\left(e^{-t^2/t_0^2}\right) \rightarrow t_0 e^{-\frac{1}{4}\omega^2 t_0^2}$$

- Pulse duration:  $t_0$
- Spectral width (bandwidth):  $\delta\omega = 2/t_0$
- Time-bandwidth product:  $t_0 \delta\omega = 2$

- This relation depends on how widths are defined

- Here we've been using 1/e half width in the field

- For FWHM in intensity:  $E(t) = E_0 e^{-2 \ln 2 t^2 / \tau^2} \rightarrow I(t) \propto e^{-4 \ln 2 t^2 / \tau^2}$

$$\tau = t_0 \sqrt{2 \ln 2} \quad \Delta\omega = \delta\omega \sqrt{2 \ln 2}$$

$$t_0 \delta\omega = 2 = \frac{\tau \Delta\omega}{2 \ln 2} \rightarrow \tau \Delta\omega = 4 \ln 2 \approx 2.77$$

$$\tau \Delta\nu = \frac{4 \ln 2}{2\pi} \approx 0.44$$

# FT(rect)=sinc and Dirac delta

- Rect( $t/t_0$ )  $\text{rect}\left(\frac{t}{t_0}\right) = 1$  for  $|t| < \frac{t_0}{2}$

$$F(\omega) = \int_{-\infty}^{\infty} \text{rect}\left(\frac{t}{t_0}\right) e^{i\omega t} dt = \int_{-t_0/2}^{t_0/2} e^{i\omega t} dt = \frac{1}{i\omega} (e^{i\omega t_0/2} - e^{-i\omega t_0/2}) \\ = t_0 \frac{\sin(\omega t_0/2)}{\omega t_0/2} = t_0 \text{sinc}(\omega t_0/2)$$

- Dirac delta  $\int_{-\infty}^{\infty} \delta(t) dt = 1$

– Limit:

$$\delta(\omega) = \lim_{t_0 \rightarrow \infty} FT \left\{ \text{rect}\left(\frac{t}{t_0}\right) \right\} = \lim_{t_0 \rightarrow \infty} [t_0 \text{sinc}(\omega t_0/2)]$$

– At  $\omega=0$ , limit is  $\infty$

–  $\omega \neq 0$ , limit is 0 in sense that integral over rapid osc  $\sin(\omega t)$  is 0

– Normalization:

$$FT \{1\} = 2\pi\delta(\omega)$$

$$FT^{-1} \{1\} = \delta(t)$$

# FT theorems

## Properties of Fourier Transforms

$A_1$  and  $A_2$  arbitrary constants

$x_0$  and  $\xi_0$  real constants

$b$  and  $d$  real nonzero constants

$k$  a positive integer

$$g(x) = \int_{-\infty}^{\infty} G(\beta) e^{j2\pi\beta x} d\beta$$

$$G(\xi) = \int_{-\infty}^{\infty} g(\alpha) e^{-j2\pi\alpha\xi} d\alpha$$

$$f(\pm x)$$

$$F(\pm \xi)$$

$$f^*(\pm x)$$

$$F^*(\mp \xi)$$

$$F(\pm x)$$

$$f(\mp \xi)$$

$$F^*(\pm x)$$

$$f^*(\pm \xi)$$

$$f\left(\frac{x}{b}\right)$$

$$|b|F(b\xi)$$

scaling

$$|d|f(dx)$$

$$F\left(\frac{\xi}{d}\right)$$

$$f(x \pm x_0)$$

$$e^{\pm j2\pi x_0 \xi} F(\xi)$$

shift

$$e^{\pm j2\pi \xi_0 x} f(x)$$

$$F(\xi \mp \xi_0)$$

# FT Theorems

Shift theorem

$$\Im\{f(t - t_0)\} = \exp(+i\omega t_0) F(\omega) \quad \Im^{-1}\{F(\omega - \omega_0)\} = \exp(-i\omega_0 t) f(t)$$

Scale theorem

$$\Im\{f(at)\} = \frac{1}{|a|} F(\omega/a) \quad \Im^{-1}\{F(b\omega)\} = \frac{1}{|b|} f(t/b)$$

Conjugation

$$\Im\{f^*(t)\} = F^*(-\omega)$$

# Symmetry properties of FT

## Symmetry Properties of Fourier Transforms

$f(x)$	$F(\xi)$
Complex, no symmetry	Complex, no symmetry
Hermitian	Real, no symmetry
Antihermitian	Imaginary, no symmetry
Complex, even	Complex, even
Complex, odd	Complex, odd
Real, no symmetry	Hermitian
Real, even	Real, even
Real, odd	Imaginary, odd
Imaginary, no symmetry	Antihermitian
Imaginary, even	Imaginary, even
Imaginary, odd	Real, odd