MATH348: SPRING 2012 - HOMEWORK 7

LINEAR ALGEBRA AND ITS APPLICATIONS

So now, less than five years later, you can go up on a steep hill in Las Vegas and look west, and with the right kind of eyes you can almost see the high water mark – that place where the wave finally broke and rolled back

ABSTRACT. Broadly speaking, linear algebra is the study of finite dimensional vector spaces, which is to say that a primary goal is to find the coefficients in the linear combination,

$$\sum_{i=1}^{n} x_i \mathbf{a}_i, \quad x_i \in \mathbb{R},\ \mathbf{a}_i \in \mathbb{R}^m, \tag{1}$$

where the upper bound of the sum is decidedly finite as opposed to the case of Fourier series or solutions to linear PDE. This topic has been studied, in one form or another, throughout human history but the context of vector spaces began in the late 1800's and from this the theory of linear transformations of finite-dimensional vector spaces emerged at the turn of the century. As opposed to the topics we have considered thus far, the theory of linear algebra is about as complete as one could hope. That is, we know about as much as there is to know, algebraically and geometrically, about finitely many linear objects of finitely many unknowns. For our study we consider the following algorithms of linear algebra:

1. Row operations and the row reduction algorithm
2. Matrix multiplication
3. Determinants of square matrices

It turns out that you have seen each of these algorithms in the math you have studied thus far. The point of linear algebra is to generalize them to arbitrary but finite data.[0] It turns out that one can study many equations of linear algebra by understanding these two algorithms. Namely, from these methods we will be able to understand:

1. Whether a point in space $\mathbf{b}$ can be written as the linear combination of directions $\mathbf{a}_i$ for $i = 1, 2, 3, \dots, n$.
2. Whether a set of flat things simultaneously intersect in space and if so, at what points?
3. Whether the solution to the matrix equation $\mathbf{Ax} = \mathbf{b}$ exists and is unique.
4. Whether the matrix $\mathbf{A}_{n\times n}$ is invertible and what the inverse is.
5. What are the special vectors $\mathbf{x} \in \mathbb{R}^n$ such that they are only scaled by the matrix multiplication $\mathbf{Ax} = \lambda\mathbf{x}$ by a factor of $\lambda \in \mathbb{C}$.
6. Knowing the eigenvalues and eigenvectors of a system, when is it possible to write the system in the diagonalized form $\mathbf{A} = \mathbf{PDP}^{-1}$?

There are, of course, more problems in linear algebra but these are those accessible in our time frame with the previous algorithms. The following list discusses how these problems are related to the previous topics.

P1. This problem is a practice in row-reduction and the conclusions one can draw from it. I ask you to take define an augmented matrix and from its row-echelon form, discuss the geometry of the linear system and the linear combination.

P2. This problem is a practice in matrix products and gives a little insight into the meaning of certain matrix multiplications through the eyes of linear transformations of the underlying vector space.

P3. This problem continues the previous discussion with the addition of matrix inversion and determinants.

P4. This problem joins the previous calculations from the point of view of an eigenvalue/eigenvector problem defined by $\mathbf{Ax} = \lambda\mathbf{x}$.

P5. It is interesting to note that a symmetric matrix has a particularly nice diagonalized form.

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[0]For comfort we note that the three algorithms correspond to:

1. Solutions Sets to Linear Systems of Algebraic Equations

Given,

$$\mathbf{A}_1 = \begin{bmatrix} 1 & -3 & 0 \\ -1 & 1 & 5 \\ 0 & 1 & 1 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 6 & 18 & -4 \\ -1 & -3 & 8 \\ 5 & 15 & -9 \end{bmatrix}, \quad \mathbf{A}_3 = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 3 \end{bmatrix}, \quad \mathbf{A}_4 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix},$$

$$\mathbf{b}_1 = \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 20 \\ 4 \\ 11 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{b}_4 = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix},$$

$$\mathbf{A}_5 = \begin{bmatrix} 5 & 3 \\ -4 & 7 \\ 9 & -2 \end{bmatrix}, \quad \mathbf{b}_5 = \begin{bmatrix} 22 \\ 20 \\ 15 \end{bmatrix},$$

$$\mathbf{A}_6 = \begin{bmatrix} 5 & 3 \\ -4 & 7 \\ 9 & -2 \end{bmatrix}, \quad \mathbf{b}_6 = \begin{bmatrix} 22 \\ 20 \\ 15 \end{bmatrix},$$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -5 \\ 7 \\ 8 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ h \end{bmatrix},$$

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} -3 \\ 9 \\ -6 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} 5 \\ -7 \\ h \end{bmatrix},$$

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix},$$

$$\mathbf{A}_7 = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}, \quad \mathbf{b}_7 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}.$$

1.1. **Algebra.** Find all solutions to $\mathbf{A}_i\mathbf{x} = \mathbf{b}_i$ for $i = 1, 2, 3, 4, 5, 6, 7$.

1.2. **Geometry.** Describe or plot the geometry formed by the linear systems and their solution sets.

1.3. **Linear Combinations.** Which of the vectors, $\mathbf{b}_i$ for $i = 1, 2, 3, 4, 5, 6, 7$, can be written as a linear combination of the columns of $\mathbf{A}_i$ for $i = 1, 2, 3, 4, 5, 6, 7$.

1.4. **Extra Credit: Linear Dependence.** Determine all values for h such that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ forms a linearly dependent set.[1]

1.5. **Extra Credit: Linear Independence.** Determine all values for h such that $S = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ forms a linearly independent set.

1.6. **Extra Credit: Spanning Sets.** How many vectors are in $S = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$? How many vectors are in span(S)? Is $\mathbf{y} \in$span(S)?[2]

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i. Row reduction is the codification of the high-school algebra applied to systems of equations but now represented in matrix form.
ii. Matrix multiplication is built off of the standard scalar-product introduced in calculus and physics.
iii. The cross-product was a kind of symbolic determinant, which can be, in some sense, generalized.

[1]Recall that vectors are considered linearly independent if and only if $c_1 = c_2 = c_3 = \cdots = c_n = 0$ is the only solution to $\sum_{i=1}^{n} c_i\mathbf{v}_i = \mathbf{0}$.

[2]Span is a notation for the set of all linear combinations that can be made from a set of vectors. That is,

$$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \ldots, \mathbf{v}_n\} = \left\{\mathbf{y} : \mathbf{y} = \sum_{i=1}^{n} c_i\mathbf{v}_i, \text{ for } c_i \in \mathbb{R}, i = 1, 2, 3, \ldots, n\right\} \tag{2}$$

1.7. **Extra Credit: Matrix Spaces.** Is $\mathbf{b}_2 \in \text{Nul}(\mathbf{A}_2)$? Is $\mathbf{b}_2 \in \text{Col}(\mathbf{A}_2)$? [3]

2. Rotation Transformations in $\mathbb{R}^2$ and $\mathbb{R}^3$

Given,

$$\mathbf{A}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

2.1. **The Unit Circle.** Show that the transformation $\mathbf{A}\hat{\mathbf{i}}$ rotates $\hat{\mathbf{i}} = [1\ 0]^{\mathrm{T}}$ counter-clockwise by an angle θ and defines a parametrization of the *unit circle*. What matrix would undo this transformation?

2.2. **Determinant.** Show that $\det(\mathbf{A}) = 1$.

2.3. **Inverse Transformation.** Find a formula for $\mathbf{A}^{-1}$. [4] Describe the geometric transformation embodied by $\mathbf{A}^{-1}$.

2.4. **Orthogonality.** Show that $\mathbf{A}^{-1} = \mathbf{A}^{\mathrm{T}}$ where $[\mathbf{A}^{\mathrm{T}}]_{ij} = \mathbf{A}_{ji}$.

2.5. **Rotations in $\mathbb{R}^3$.** Given,

$$\mathbf{R}_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \quad \mathbf{R}_2(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \quad \mathbf{R}_3(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Describe the transformations defined by each of these matrices on vectors in $\mathbb{R}^3$.

3. Square Coefficient Data, Matrix Inversion and Determinants

Given,

$$\mathbf{A} = \begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{bmatrix}.$$

3.1. **Matrix Inverse: Take One.** Find $\mathbf{A}^{-1}$ using the Gauss-Jordan Method. (pg.317)[5]

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[3] The following are definitions for each space:

$$\text{Nul}(\mathbf{A}) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = \mathbf{0}, \mathbf{A} \in \mathbb{R}^{m\times n}\} \tag{3}$$

$$\text{Col}(\mathbf{A}) = \left\{\mathbf{y} \in \mathbb{R}^m : \mathbf{y} = \sum_{i=1}^{n} c_i\mathbf{a}_i,\ c_i \in \mathbb{R}\right\} \tag{4}$$

We would say that the *null-space*, Nul($\mathbf{A}$), is the set of all solutions to the homogeneous problem $\mathbf{A}\mathbf{x} = \mathbf{0}$ and tells us about the geometry of intersection between the many linear objects. Since this is the homogeneous problem, it always has a solution, the question the null-space addresses is whether there are other points of intersection. In a similar vein, the *column-space*, Col($\mathbf{A}$), is the set off all vectors that can be made using the columns of the original matrix. Essentially, these spaces tell us the following:

i. Null-space: If the linear objects are forced to intersect then what is the intersection geometry.
ii. Column-space: What are the origin offsets permitted such that the previous intersection geometry holds, up to translation away from the origin.

Lastly, to find out whether a vector is in the null-space simply see if the vector satisfies $\mathbf{A}\mathbf{x} = \mathbf{0}$. On the other hand to see if a vector is in the column-space you must consider whether $\mathbf{A}\mathbf{x} = \mathbf{b}$ is consistent.

[4] You may want to remember that $\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

[5] Since we don't have a book I will say that the Gauss-Jordan method is the row-reduction method where you attempt $[\mathbf{A}|\mathbf{I}] \sim [\mathbf{I}|\mathbf{A}^{-1}]$. If you cannot make this happen then there is no inverse matrix.

3.2. **Matrix Inverse: Take Two.** Find $\mathbf{A}^{-1}$ using the cofactor representation. (Theorem 2 pg.318)[6]

3.3. **Solutions to Linear Systems.** Using $\mathbf{A}^{-1}$ find the unique solution to $\mathbf{Ax} = \mathbf{b} = [b_1\, b_2\, b_3]^{\mathrm{T}}$.

Given,

$$\mathbf{A} = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}.$$

3.4. ***Vandermonde Determinant.*** [7] Show that the $\det(\mathbf{A}) = (c-a)(c-b)(b-a)$.

3.5. **Application.** Determine which of the following sets of points can be uniquely interpolated by the polynomial $p(t) = a_0 + a_1 t + a_2 t^2$. [8]

$$S_1 = \{(1,12),(2,15),(3,16)\}$$
$$S_2 = \{(1,12),(1,15),(3,16)\}$$
$$S_3 = \{(1,12),(2,15),(2,15)\}$$

4. Eigenvalues and Eigenvectors

$$\mathbf{A}_1 = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}, \quad \mathbf{A}_3 = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}, \quad \mathbf{A}_4 = \begin{bmatrix} .1 & .6 \\ .9 & .4 \end{bmatrix}, \quad \mathbf{A}_5 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},$$

4.1. **Eigenproblems.** Find all eigenvalues and eigenvectors of $\mathbf{A}_i$ for $i = 1,2,3,4,5$.

4.2. **Diagonlization.** Find all matrices associated with the diagonalization of $\mathbf{A}_i$ for $i = 3,4,5$.

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[6] This problem is just more work with determinants. We remember from lecture that determinants can be taken down any row or column of the matrix. Specifically,

$$\det(\mathbf{A}) = \sum_{\substack{i=1,j \text{ fixed} \\ \text{or} \\ j=1,i \text{ fixed}}}^{n} a_{ij}(-1)^{i+j}\det(\mathbf{A}_{ij}) = \sum_{\substack{i=1,j \text{ fixed} \\ \text{or} \\ j=1,i \text{ fixed}}}^{n} a_{ij}C_{ij}, \tag{5}$$

where $C_{ij} = (-1)^{i+j}\det(\mathbf{A}_{ij})$ is called the $(i,j)-$cofactor of $\mathbf{A}$, which contains the determinant of the $(i,j)-$minor of $\mathbf{A}$, $\mathbf{A}_{ij}$. The minor matrix is found by removing the $i^{th}-$row and $j^{th}-$column of $\mathbf{A}$. It turns out that from this, the inverse of a matrix has a closed form with elements given by

$$\left[\mathbf{A}^{-1}\right]_{ij} = \frac{1}{\det(\mathbf{A})}C_{ji}. \tag{6}$$

Such formulae might be useful if one wanted specific elements out of an inverse matrix and not necessarily the whole thing, which is what you would have to do to find the inverse via Gauss-Jordan.

[7] This matrix is common in the study of polynomial interpolation.

[8] I always think this problem is neat. First, we are going to use linear algebra on a nonlinear polynomial. This shouldn't trouble you much b/c we did the same thing with Fourier series. Remember so long is the equation is a 'vector' times a constant plus more of the same, then we are talking about a linear problem. In this case the 'vectors' are power functions and the scalars are what we are looking for. So, we consider the first set of points. If the polynomial p is to satisfy these points then we must have

$$a_0 + a_1 + a_2 = 12 = p(1)$$
$$a_0 + 2a_1 + 4a_2 = 15 = p(2)$$
$$a_0 + 3a_1 + 9a_2 = 16 = p(3),$$

which is a linear system of equation where the unknowns are the a_i's. Now the question is does there exist a solution and is the solution unique. If so then there is a set of coefficients so that the quadratic equation touches each of the points in S_1. You may also notice that the coefficient matrix defined by this problem is of the form from subsection (3.4).

4.3. **Extra Credit:** ***regular stochastic matrix.*** For $\mathbf{A}_4$ define its associated steady-state vector, $\mathbf{q}$, to be such that $\mathbf{A}_4\mathbf{q} = \mathbf{q}$.[9]

4.4. **Extra Credit: Limits of Time Series.** Show that $\lim_{n\to\infty} \mathbf{A}_4^n\mathbf{x} = \mathbf{q}$ where $\mathbf{x} \in \mathbb{R}^2$ such that $x_1 + x_2 = 1$.

5. Extra Credit: Orthogonal Diagonalization and Spectral Decomposition

Recall that if $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ then their inner-product is defined to be $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x}^{\mathrm{H}}\mathbf{y} = \bar{\mathbf{x}}^{\mathrm{T}}\mathbf{y}$. In this case, the 'length' of the vector is $|\mathbf{x}| = \sqrt{\langle \mathbf{x}, \mathbf{x}\rangle}$. [10]

5.1. **Self-Adjointness.** Show that $\mathbf{A}_5$ is a self-adjoint matrix.[11]

5.2. **Orthogonal Eigenvectors.** Show that the eigenvectors of $\mathbf{A}_5$ are orthogonal with respect to the inner-product defined above.

5.3. **Orthonormal Eigenbasis.** Using the previous definition for length of a vector and the eigenvectors of the self-adjoint matrix, construct an *orthonormal basis* for $\mathbb{C}^2$.

5.4. **Orthogonal Diagonalization.** Show that $\mathbf{U}^{\mathrm{H}} = \mathbf{U}^{-1}$, where $\mathbf{U}$ is a matrix containing the normalized eigenvectors of $\mathbf{A}_5$.

5.5. **Spectral Decomposition.** Show that $\mathbf{A}_5 = \lambda_1\mathbf{x}_1\mathbf{x}_1^{\mathrm{H}} + \lambda_2\mathbf{x}_2\mathbf{x}_2^{\mathrm{H}}$.

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[9] A regular stochastic matrix is a matrix whose elements sum to one for each column and represents the probability of going from one state, say $\mathbf{x}_0$, to another state, say $\mathbf{x}_1$, after a multiplication. That is, if $\mathbf{A}$ is a regular stochastic matrix then we have $\mathbf{A}\mathbf{x}_0 = \mathbf{x}_1 \implies \mathbf{A}^2\mathbf{x}_0 = \mathbf{x}_2$ and generally $\mathbf{A}^n\mathbf{x}_0 = \mathbf{x}_n$. The question that is typically asked at this point is whether this process limits to some steady-state vector. In this first question seeks to find the vector that is unchanged by the multiplication by $\mathbf{A}_4$. The next question will show this vector is the limit of infinitely-many applications of the matrix $\mathbf{A}_4$, which can only really be found via eigenvalues/eigenvectors and diagonalization.

[10] First let's remember that $[\mathbf{A}^{\mathrm{T}}]_{ij} = \mathbf{A}_{ji}$ is the transpose of a matrix and just tells us to swap rows with columns. It is interesting to note that the scalar product can then be rewritten in terms of a matrix transpose. Consider the scalar product, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ then

$$\mathbf{x}\cdot\mathbf{y} = \mathbf{x}^{\mathrm{T}}\mathbf{y} = \sum_{i=1}^{n} x_i y_i, \tag{7}$$

which can thus be written as using matrix multiplication and matrix transposition. If $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$, which are vectors whose elements are complex then the scalar product is now given by the adjoint. That is,

$$\mathbf{x}\cdot\mathbf{y} = \mathbf{x}^{\mathrm{H}}\mathbf{y} = \sum_{i=1}^{n} \bar{x}_i y_i, \tag{8}$$

which now gives a way to measure the angle between two complex vectors.

[11] A matrix adjoint means two things:

1. Take the transpose of the matrix.
2. Take the complex conjugate of each element in the matrix.

and is the analogy of a transposition for matrices with complex entries. It turns out the every matrix that equals its own adjoint has a very special diagonal form. We find this form in these problem.