E. Kreyszig, Advanced Engineering Mathematics, 9^{th} ed.	Section 12.1, pgs. 535-538
Lecture: Introduction to PDE	Module: 13
Suggested Problem Set: {17, 19, 22, 23, 24, 26(c), 27)}	Last Compiled: April 12, 2010

Quote of Lecture 13	
If you didn't care what happened to me and I didn't care for you. We would zig-zag our way through the boredom and pain occasionally glancing up through the rain. Wondering which of the buggers to blame and watching for pigs on the wing.	
	Pink Floyd : Pigs on the Wing - Part 1 (1977)

At long last we start our study of partial differential equations (PDE). We will see everything we have studied this semester come back again as we learn the classical methods of solving *linear* PDE. The emphasis here is on the linearity of the PDE. Without this the following important tools,

- Linear combinations of basis vectors
- Eigenvalues and eigenvectors
- General solution to linear ODE
- Fourier series/integral

are rendered almost useless. [1] However, before we begin that discussion it makes sense to discuss some of the basic terminology.

Definition 1. Linearity of an Equation - We say that an equation, differential or otherwise, is linear in some quantity if it can be written as a lin ear combination of the quantity. In the case of a PDE the quantity is the unknown function u, which may depend on many variables say x, y, z, t. The PDE is then linear of it can be written as a linear combination of u and derivatives of u. The general notation can be cumbersome so this is best illustrated by examples.

Example 1. *Linear PDE - The following are some examples of common linear PDE:*[2]

(1) $\triangle u = 0.$ [3]
(2) $\triangle u = f(x, y, z).$ [4]
(3) $c^2 \triangle u = \dfrac{\partial u}{\partial t}.$ [5]
(4) $c^2 \triangle u = \dfrac{\partial^2 u}{\partial^2 t}.$ [6]
(5) $\dfrac{\partial u}{\partial t} = c\dfrac{\partial u}{\partial t}.$ [7]

Example 2. *Nonlinear PDE - The following are some examples of common nonlinear PDE:*

(1) $\rho \left(\dfrac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} \right) = -\nabla p + \nu \triangle \boldsymbol{v}.$ [8]

[1] I say almost here because to understand nonlinear theory, which is at this time woefully incomplete, one must understand the completeness of linear theory. So, these principles come back again to study nonlinear theory but don't often tell you everything you want to know.

[2] In the following we use notations common in vector calculus. For a review of these notations please see homework 7.

[3] This equation is called Laplaces equation and models space subjected to potential fields like gravitational or electrostatic.

[4] This equation is called Possions equation and models space subjected to potential fields with source terms.

[5] This equation is called the heat or diffusion equation and models the time-dynamics of the flow of a density, which tends to move from areas of high density to low density.

[6] This equation is called the wave equation and models the standing waves or traveling waves in an elastic medium.

[7] This equation is called a convection equation or transport equation and models the pure transport of a material due to movements of its background medium.

[8] This equation is called the Navier-Stokes equation and can be derived as the model equation for the evolution of fluid particles. See also *wikipedia* and *Clay Math*.

(2) $i\dfrac{\partial u}{\partial t} = c^2\triangle u + \lambda|u|^2 u.$ [9]

(3) $\dfrac{\partial u}{\partial t} + \dfrac{\partial^3 u}{\partial^3 x} + 6u\dfrac{\partial u}{\partial x} = 0.$ [10]

Remark 1. *The critical point here is the for a PDE to be linear you cannot have terms like,*

$$u^2, \quad \sin(u), \quad u\frac{\partial u}{\partial x}, \quad \left(\frac{\partial^2 u}{\partial t^2}\right)^2, \tag{0.1}$$

while terms like,

$$u, \quad \sin(x)u, \quad xy\frac{\partial u}{\partial x}, \quad t^2\frac{\partial^2 u}{\partial t^2}, \tag{0.2}$$

are permitted.

Definition 2. Homogeneous PDE - We say that a PDE is homogeneous if it is linear and does not contain terms where the dependent variable u or derivatives of this unknown function are absent. If the linear PDE contains a term, which does not depend on the unknown function or its derivative then we say that the PDE is inhomogeneous. Of the previous linear examples, (x), (x) and (x), are homogeneous, while the others are not. If you set $F(x, y, z, t) = 0$ in the remaining examples then they would be homogeneous equations.

Theorem 1. *Superposition of Solutions - If a PDE is linear and homogeneous and u_1 and u_2 are solutions to this PDE then the arbitrary linear combination $u = a_1u_1 + a_2u_2,\ a_1, a_2 \in \mathbb{R}$ is also a solution.*

Proof. In general, since the derivative of a sum is the sum of derivatives superpositions will be decomposed by the PDE into 'smaller' equivalent PDE. If each term in the superposition is a solution then each 'smaller' equivalent PDE is subsequently satisfied. Specifically, for the heat equation, we have,

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial}{\partial t}\left[a_1u_1 + a_2u_2\right] && (0.3)\\
&= a_1\frac{\partial u_1}{\partial t} + a_2\frac{\partial u_2}{\partial t} && (0.4)\\
&= a_1c^2\triangle u_1 + a_2c^2\triangle u_2 && (0.5)\\
&= \triangle\left[a_1u_1 + a_2u_2\right] && (0.6)\\
&= c^2\triangle u. && (0.7)
\end{aligned}$$

These arguments hold for any homogeneous linear equation. □

This, coupled with all of your mathematics up to now, completes the basic background necessary for the study of linear PDE, which will begin with a derivation of the so-called heat or diffusion equation. The heat/diffusion equation is a first-order in time and second-order in space linear PDE on $\mathbb{R}^{3+1}$ and models the time-dynamics of a conserved *density* whose flow is along its spatial gradient. In this derivation we focus on the generality of conservation principles and their closure with constitutive relations.

The wave equation on $\mathbb{R}^{1+1}$ can be derived in the context of a vibrating ideally elastic string by analysis of force at a point on the string.[11] This can also be done for in $\mathbb{R}^{2+1}$ for a thin ideally elastic membrane.[12] However, the wave equation can manifest more generally in the context of the Einstein field equations on a vacuum background or in terms of small disturbances of any elastic background medium. Though, this is outside the scope of our course it is interesting to know since our characterizations of solutions to the wave equation will hold in all of these contexts and give insight into how nonlinearity might appear. Beginning with a vibrating elastic rectangular membrane we will derive the solution to the wave equation on a closed rectangular subset of $\mathbb{R}^{2+1}$ by the use of double Fourier series and moving to a vibrating circular membrane we will see how the solution allows for more complicated vibrational modes, which can be interpreted physically in terms of musical instruments. Lastly, we characterized solutions to the wave equation in terms of traveling

[9]This equation is called the nonlinear Schrödinger equation and models the evolution of a new phase of matter called a Bose-Einstein condensate whose 1995 experimental observation earned researchers at Boulder and MIT and Nobel prize in 2001! See also *CU Boulder* and *wikipedia*

[10]This equation is called the Kortewegde Vries equation an models surface waves in shallow waters and was the basis for modern advances in the study of exactly solvable nonlinear PDE. *Wikipedia KdV*

[11]See EK section 12.2 for details.

[12]See EK section 12.7 for details.

waves whose speed and influence can be determined by the concept of characteristics, which gives rise to sounds-speeds and the speed-of-light.

1. Lecture Goals

Our goal with this material will be:

- Understand the mathematical definition of PDE as well as some of their modeling capabilities.

2. Lecture Objectives

The objectives of these lessons will be:

- List various PDE and their associated models.
- Define the vocabulary associated with PDE with an emphasis on the interplay between linearity and superposition.
- Outline direction and key points of our study of PDE.