MATH332-Linear Algebra	**Homework Five**

Eigenproblems, Diagonalization and Applications

Text: 5.1-5.3, 5.5 Section Overviews: 5.1-5.3, 5.5

Quote of Homework Five	
Limber Up	
	Zombieland : Rule #18 (2009)

1. Eigenvalues and Eigenvectors

$$\mathbf{A}_1 = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}, \quad \mathbf{A}_3 = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}, \quad \mathbf{A}_4 = \begin{bmatrix} .1 & .6 \\ .9 & .4 \end{bmatrix}, \quad \mathbf{A}_5 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},$$

1.1. **Eigenproblems.** Find all eigenvalues and eigenvectors of $\mathbf{A}_i$ for $i = 1, 2, 3, 4, 5$.

2. Applications of Diagonalization

2.1. **Eigenbasis and Decoupled Linear Systems.** Find the diagonal matrix $\mathbf{D}_i$ and vector $\tilde{\mathbf{Y}}$ that completely decouples the system of linear differential equations $\frac{d\mathbf{Y}_i}{dt} = \mathbf{A}_i\mathbf{Y}_i$ for $i = 3, 4, 5$.

2.2. **Regular Stochastic Matrices.** For the *regular stochastic matrix* $\mathbf{A}_4$, define its associated steady-state vector, $\mathbf{q}$, to be such that $\mathbf{A}_4\mathbf{q} = \mathbf{q}$. Show that $\mathbf{q} = [2/5\ 3/5]^{\mathrm{T}}$.

2.3. **Limits of Time Series.** Show that $\lim_{n\to\infty} \mathbf{A}_4^n\mathbf{x} = \mathbf{q}$ where $\mathbf{x} \in \mathbb{R}^2$ such that $x_1 + x_2 = 1$.

3. Theoretical Results

3.1. **Spectrum of Self-Adjoint Matrices.** Show that if $\mathbf{A} = \mathbf{A}^{\mathrm{H}}$ then $\sigma(\mathbf{A}) \subset \mathbb{R}$.

3.2. **Connection to Invertible Matrices.** Show that if $\mathbf{A}$ is both diagonalizable and invertible then so is $\mathbf{A}^{-1}$.

3.3. **Connection to Transposition.** Show that if $\mathbf{A}$ has $n-$many linearly independent eigenvectors then so does $\mathbf{A}^{\mathrm{T}}$.

4. A Taste of Things to Come

Of the previous matrices only one of them has a direct relation to its Hermitian-adjoint. Recall, that in homework 2 we found that $\mathbf{A}_5^{\mathrm{H}} = \bar{\mathbf{A}}_5^{\mathrm{T}} = \mathbf{A}_5$ and that we called a matrix with this property self-adjoint. The following set of problems shows exactly how nice self-adjoint matrices are.

4.1. **Dot-Products Redux.** Recall that if $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ then we define their dot-product as $\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 \in \mathbb{R}$ and that this gave us some information about the angle between the two vectors. This formula is the same as the matrix-product $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^{\mathrm{T}}\mathbf{y}$ but if you apply it to the eigenvectors from $\mathbf{A}_5$ it will return non-sensible results. [1] The problem is that the vectors are not from $\mathbb{R}^2$ but are from $\mathbb{C}^2$. This problem is resolved by the Hermitian-adjoint. That is, whenever vectors are from $\mathbb{C}^2$ the dot-product is defined by, $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^{\mathrm{H}}\mathbf{y}$, which returns sensible results.[2] [3] With that said, show that the eigenvectors for $\mathbf{A}_5$ are orthogonal.

4.2. **Orthonormality.** Using this definition of dot-product scale both eigenvectors to be unit length.

4.3. **Unitarity.** Using the normalized eigenvectors construct the matrix $\mathbf{U} = [\hat{\mathbf{x}}_1\ \hat{\mathbf{x}}_2]$ and show that $\mathbf{U}^{\mathrm{H}}\mathbf{U} = \mathbf{I}$.

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[1] If you want to, try it! Consider $\mathbf{x}_1 \cdot \mathbf{x}_1 = \mathbf{x}_1^{\mathrm{T}}\mathbf{x}_1 = 0$ seems to imply that the vector has no length. This is a problem.

[2] From this you will find that $\mathbf{x}_1 \cdot \mathbf{x}_1 = \mathbf{x}_1^{\mathrm{H}}\mathbf{x}_1 = 2$, which implies that the length of the vector is $\sqrt{2}$. This makes more sense since we can think of this vector as pointing one-unit in both the real and imaginary directions, which creates a $1, 1, \sqrt{2}$ triangle.

[3] This might seem crazy but it is comforting to note that if the vectors are real then $\mathbf{x}^{\mathrm{H}} = \mathbf{x}^{\mathrm{T}}$ so this is merely and abstraction for complex number systems.

4.4. **Orthogonal Diagonalization.** It will later be shown that if a matrix is self-adjoint then it always provides enough eigenvectors for diagonalization and that these vectors can be chosen to be orthonormal. Moreover, a square matrix with orthonormal columns is called unitary and will always satisfy the property from the previous subsection. This effectively removes the need for inverse computations. Show this for our special matrix by verifying that $\mathbf{A}_5 = \mathbf{U}\mathbf{D}\mathbf{U}^{\mathrm{H}}$.

4.5. **Spectral Representation.** This sort of decomposition has many applications and one is the so-called spectral representation of self-adjoint matrices. Show that $\mathbf{A}_5 = \lambda_1 \hat{\mathbf{x}}_1 \hat{\mathbf{x}}_1^{\mathrm{H}} + \lambda_2 \hat{\mathbf{x}}_2 \hat{\mathbf{x}}_2^{\mathrm{H}}$.

5. Lecture Appreciation

In lecture we considered the following applications of linear algebra:

- Ridged geometric preserving transformations of $\mathbb{R}^n$ as it relates to ridged body and fluid dynamics.
- Normal mode analysis of vibrational systems.
- Numerical approximation of solutions to partial differential equations.
- General solutions to constant linear ordinary differential equations.
- Quantum bits, Lie structures and the vector-space $\mathbb{C}^2$.

Pick one of the topics and:

(1) In five paragraphs or less, summarize the lecture.
(2) Address as many questions raised in lecture as possible.
(3) List the remaining questions that you have.