MATH 235 - Differential Equations w/ Honors March 17, 2008
Homework 5, Spring 2008 **Due**: March 27, 2008

TWO-DIMENSIONAL AUTONOMOUS LINEAR SYSTEMS OF ODE'S - BIFURCATIONS
TRACE DETERMINANT PLANE - MASS-SPRING SYSTEMS

1. Given the following coefficient matrices,

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}, \; \mathbf{A}_2 = \begin{bmatrix} 0 & 4 \\ -3 & -13 \end{bmatrix}, \; \mathbf{A}_3 = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}, \; \mathbf{A}_4 = \begin{bmatrix} 0 & 4 \\ -5 & -4 \end{bmatrix}, \; \mathbf{A}_5 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \; \mathbf{A}_6 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad (1)$$

for two-dimensional autonomous linear systems of ordinary differential equations, $\mathbf{Y}' = \mathbf{A}_i\mathbf{Y}, \;\; i = 1, 2, 3, 4, 5,$:

   (a) Calculate all eigenvalues and corresponding real-valued eigenfunctions.
   (b) Show that eigenfunctions are linearly-independent. [1]
   (c) Construct the general solution and determine the unique trajectory passing through $\mathbf{Y}(0) = (1, 1)$.
   (d) Using the program HPGSYSTEMSOLVER, associated with the text, plot the vector field, the previous trajectory, a few others, and classify the fixed point(s).

2. Given the following coefficient matrices,

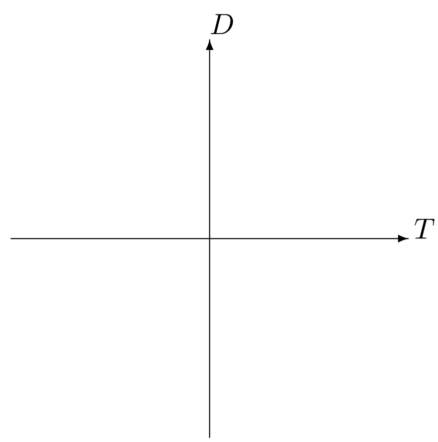
$$\mathbf{A}_1 = \begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 1 & \alpha \\ 1 & 1 \end{bmatrix}, \quad \mathbf{A}_3 = \begin{bmatrix} 1 & -\alpha \\ \alpha & 1 \end{bmatrix}, \quad (2)$$

for two-dimensional autonomous linear systems of ordinary differential equations, $\mathbf{Y}' = \mathbf{A}_i\mathbf{Y}, \;\; i = 1, 2, 3$. Determine any changes to behavior of the fixed point behavior as the parameter α is continuously changed. List any bifurcation values, if encountered, and classify the behavior of the fixed point at the bifurcation value and its local neighborhood.

3. Continue the arguments from class associated with the general two-dimensional autonomous system of linear ODE's,

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \quad (3)$$

and classify the fixed points of (3) found in the second-quadrant of the trace-determinant plane.[2] by Graph all of your information in the trace-determinant plane below and, lastly, using the program TDANIMATION discuss the changes to both the number and classification of equilibrium points as the TD-plane is traversed by the program.



[1] We can show that the eigenfunctions are linearly independent if the Wronskian matrix $\mathbf{W}(t) = \begin{bmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{bmatrix}$ has a nonzero determinant.

[2] Recall the definition of trace and determinate are given as, $\text{trace}(\mathbf{A}) = \text{tr}(\mathbf{A}) = T = a + d$, and, $\det(\mathbf{A}) = D = ad - bc$, where $\mathbf{A}$ is given by the coefficient matrix in (3).

4. Consider the model equation for a mass suspended from an ideal spring. If we include the effects of frictional forces and an external applied force, $f(t)$, we can derive from force laws[3] the second-order linear ordinary differential equations with constant coefficients:

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = f(t), \quad m, b, k \in R^+ \cup \{0\}, \tag{4}$$

   (a) Convert the second-order linear ODE (4) to a system of first-order ODE's.
   (b) If $f(t) = 0$ for all t and $b = 0$ we call this unforced oscillator *simple*. Show that the fixed point of an unforced simple harmonic oscillator is always a center.[4]
   (c) We now consider the effects of friction using MASSSPRING and the systems defined by $m = k = 1$ and $b_1 = 0,\ b_2 = 0.5,\ b_3 = 1,\ b_4 = 1.5,\ b_5 = 2$. For each of the previous systems plot a trajectory whose initial condition is somewhere near the center of the first quadrant and using these plots describe effects of friction on the long-term behavior to each of the trajectories. [5]

5. Now we consider the effects of external forcing on a simple harmonic oscillator.[6] Of all of the external forces to consider the most interesting involve periodic forcing. Here we consider an applied force given by $f(t) = F\cos(\omega t),\ F, \omega \in \mathbb{R}^+ \cup \{0\}$. Run the program FORCEDMASSSPRING for all permutations of the values, $F_1 = 1,\ F_2 = 2,\ \omega_1 = 0,\ \omega_2 = 0.5,\ \omega_3 = 0.75,\ \omega_4 = 1$, plotting the trajectories whose initial condition is roughly in the center of the first quadrant. Using this information respond to the following:

   (a) How does constant forcing effect the fixed point of the system? [7]
   (b) Now considering the parameter ω, for $\omega < 1$, how does oscillatory forcing effect the behavior of trajectories in phase space?[8]
   (c) Consider the case where $\omega = 0.75$ and looking at the graph of y versus t notice that the curve is an oscillatory function whose amplitude is itself also oscillating.[9] This pattern, which occurs when the frequency of forcing nears the frequency of natural oscillation, is called a beat pattern. Using `http://en.wikipedia.org/wiki/Beat_\%28acoustics\%29` explain the connection between this mass-spring phenomenon and acoustics.
   (d) Explain what occurs to the mass-spring system when $\omega = 1$ and give examples of other phenomenon, which have similar qualitative features. [10]

[3] Remember that when deriving this equation we used Hook's law, which says that in the elastic limit the restoring force is linearly proportional to the displacement/deformation. Outside of this limit the relationship becomes nonlinear and can be used to explain phenomenon like non-reversible deformations associated with large displacements.

[4] We may call this oscillator simple but it is also classic example of a conservative system. In this case it is energy, which is conserved. The notion of conserved quantities will be explored in the next homework and applied to nonlinear systems in chapter 5.3

[5] Friction is considered a dissipative effect. Normally when discussing a conservative system it is common to also discuss the effects of corresponding dissipative effects. This may not always be as simple as studying the effects of a single term in the system.

[6] What we are about to see here is so important to physical systems prone to oscillations that we will study it again in the next homework through the model equation (4) and not the displacement-velocity system found in problem (4).

[7] In mathematical terms the time-independent inhomogeneity has shifted the fixed point to be off the origin.

[8] Since the system is no longer autonomous there are no *fixed points*, however the trajectories do appear to be 'orbiting' points in phase space and one of them seems to correspond to the fixed point of part (a). That is to say, though we do not have *fixed points*, by definition, our understanding of them can be useful in describing non-autonomous cases.

[9] We say that the higher frequency oscillations are bounded by a lower frequency *envelope*. Qualitative changes to this envelope are important in the diffraction pattern of waves and as we will see, in a moment, resonance.

[10] You may want to consider the following website to guide your thoughts `http://en.wikipedia.org/wiki/Resonance`[11]

[11] I know I abuse the wiki, but it gets the job done. :)[12]

[12] I can't believe that I just footnoted a footnote. I guess I abuse those too!