MATH 235 - Differential Equations w/ Honors February 11, 2008
Homework 4, Spring 2008 **Due**: February 21, 2008

EXISTENCE AND UNIQUENESS - PHASE LINES - BIFURCATIONS - LINEARITY

1. Consider the autonomous first-order differential equation,

$$\frac{dy}{dt} = f(y), \tag{1}$$

where f is continuously differentiable.

(a) Suppose that $y_1(t)$ is a solution to (1) and that $y_1(t)$ has a local maximum at $t = t_0$. Let $y_0 = y_1(t_0)$. Show that $f(y_0) = 0$.

(b) Show that the constant function $y_2(t) = y_0$ is a solution to the differential equation (1).

(c) Show that $y_1(t) = y_0$ for all t.

Hint: For (c), what can be said about the uniqueness of solutions, y_1, y_2, to the differential equation?
Comment: A similar argument can be made where $y_1(t)$ has a local minimum at t_0. This implies that if $y(t)$ is a solution to (1), which has a local extrema, then this solution is a constant function. That is, if only equilibrium solutions have local extrema then non-equilibrium solutions must be either always increasing or always decreasing!

2. Given,

$$\frac{dy_1}{dt} = \alpha - y_1^2, \qquad \frac{dy_2}{dt} = \alpha + y_2^2, \qquad \frac{dy_3}{dt} = \alpha y_3 - y_3^3, \qquad \frac{dy_4}{dt} = (y_4^2 - \alpha)(y_4^2 - 4). \tag{2}$$

For each of the previous families of differential equations of the parameter α do the following:

(a) Calculate the bifurcation value(s) of α.

(b) Sketch phase lines for various values of α.

(c) Using this information draw a bifurcation diagram labeling the stable and unstable equilibrium solutions.

3. Determine the general solution for the following linear differential equations:

(a) $\dfrac{dy}{dt} - 2y = t^2 + 3e^t$ (b) $y' = 5y + 3e^{5t}$

(c) $y' = -3y + 2\cos(2t)$ (d) $y'' - 3y' + 2y = 0$

Hint: For (d) assume that $y(t) = e^{rt}$, $r \in \mathbb{R}$, and show that $y'' - 3y' + 2y = 0 \iff r^2 - 3r + 2 = 0$. Solve for r to find two possible solutions. In this case the general solution is $y(t) = c_1 y_1(t) + c_2 y_2(t),\ \ c_1, c_2 \in \mathbb{R}$.

4. Given the first-order linear ODE with variable coefficient $p(t)$, and inhomogeneity $g(t)$,

$$\frac{dy}{dt} + p(t)y = g(t). \tag{3}$$

It is possible to find an integrating factor $\mu(t)$ that when multiplied into the equation (3) makes the left-hand side into an exact derivative.

(a) Multiply (3) on both sides by a currently unknown function $\mu(t)$ and assuming that a suitable $\mu(t)$ can be found show that (3) can be written as, $\dfrac{d\,[\mu y]}{dt} = \mu(t)g(t)$.

(b) Now using the auxiliary equation from part (a) show that this integrating factor has the form $\mu(t) = e^{\int p(t)dt}$.

5. Use the results from problem 4 to solve the first-order linear ordinary differential equation with variable coefficients $y' + 2t^{-1}y = e^{2t}$.