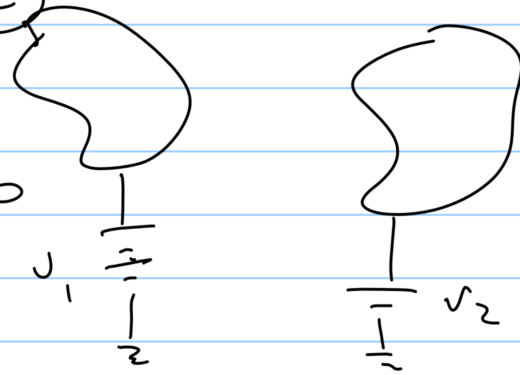
Note Title 1/28/2008

$\nabla^2 V = -\rho/\epsilon_0$

ρ

V E



Given V

$\nabla^2 V = -\rho/\epsilon_0$ in vacuum $\rho = 0$

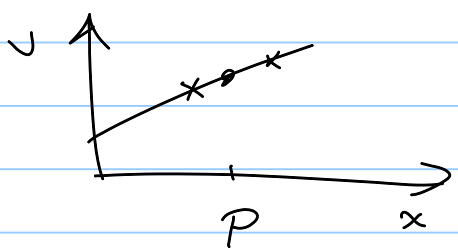
$\nabla^2 V = 0$ & bndry conditions

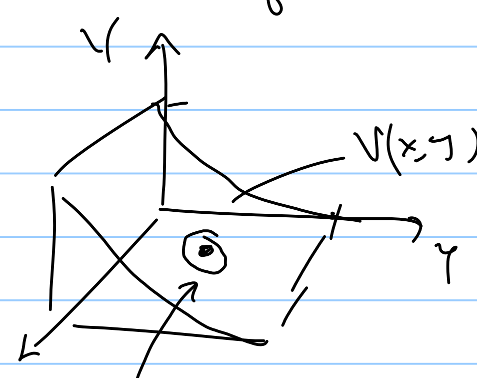
1-D $\qquad \dfrac{d^2V}{dx^2} = 0 \Rightarrow V = mx + b$

Need 2 bndry conditions: $V(x=0)$ & $V(x=L)$

2 properties of Laplaces eqn ($\nabla^2 V = 0$)

(1) V at point P is the average of V in neighborhood of P





(2) V can have no local maxima or minima (extrema are at bndries)

$$\frac{1}{2\pi R}\oint V\,dl = V_{ave}$$

In more than 1-D we need to specify V at a bndry which has ∞ # of pts.

⇒ Partial Diff Eqn much harder to solve than Ordinary Diff Eqn

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Uniqueness Thm: If V is a soln to Poisson's eqn & V is known of all boundaries
⇒ V is unique

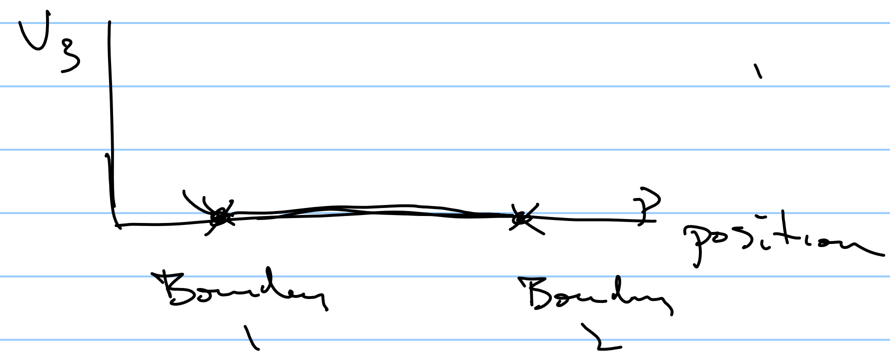
Proof: assume 2 soln V_1 & V_2. $\nabla^2 V = -\rho/\epsilon_0$ is

LINEAR. $\nabla^2 V_1 = -\rho/\epsilon_0$ $\nabla^2 V_2 = -\rho/\epsilon_0$

$V_3 = V_1 - V_2$ is also soln

$$\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = \frac{\rho}{\epsilon_0} - \frac{\rho}{\epsilon_0} = 0$$

V_3 ~~satisfies~~ satisfies Laplaces eqn



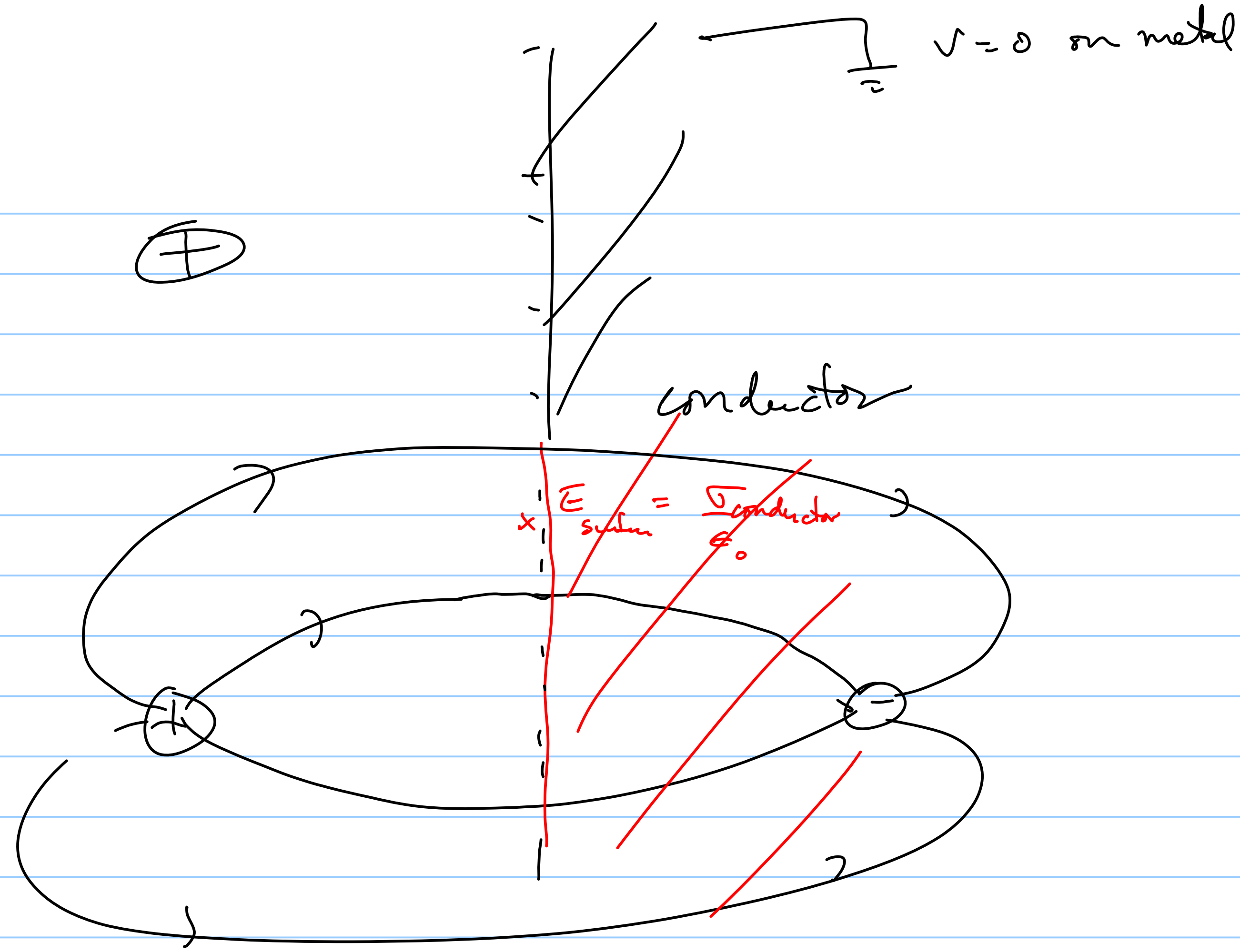
$V_1 = V_2$ at boundary

⇒ $V_3 = 0$ everywhere $V_1 = V_2$ everywhere

V is unique

If we find a soln to $\nabla^2 V = 0$ & boundary condition it is the only soln. we are done

Ex:



Solve $\nabla^2 V = 0$ by sep. variables

assume $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$

$V(x,y,z) = X(x)\,Y(y)\,Z(z)$ Not a general soln to Laplaces eqn

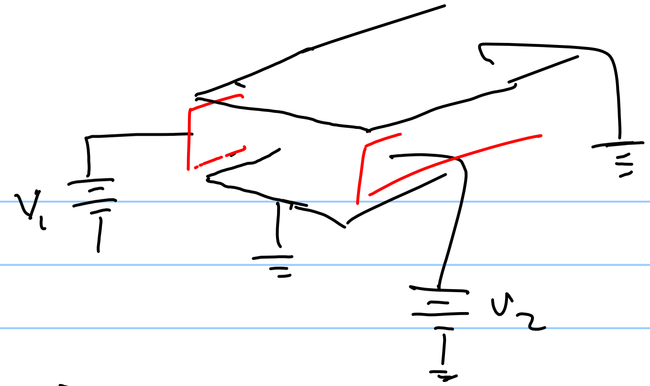
Plug in to $\nabla^2 V = 0$ & divide by $V = XYZ$

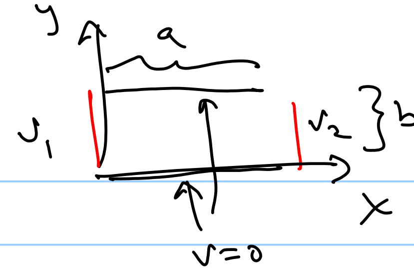
$$\underbrace{\frac{1}{X}\frac{d^2 X}{dx^2}}_{C_1} + \underbrace{\frac{1}{Y}\frac{d^2 Y}{dy^2}}_{C_2} + \underbrace{\frac{1}{Z}\frac{d^2 Z}{dz^2}}_{C_3} = 0$$

From 1 PDE to 3 ODE

$$\begin{cases} \frac{1}{X}\frac{d^2 X}{dx^2} = C_1 \\ \frac{1}{Y}\frac{d^2 Y}{dy} = C_2 \end{cases}$$

Ex:





Solve $\nabla^2 V = 0$ inside box & get $\vec{E} = -\vec{\nabla} V$ inside

Then $E_\perp$ near the boundaries to get σ from $E_\perp = \frac{\sigma}{\epsilon_0}$

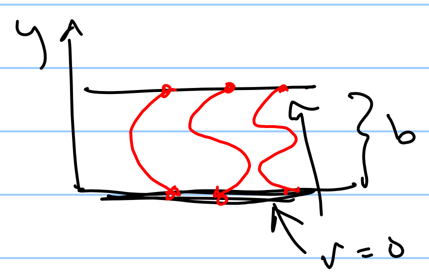
$$\text{B.C.}\begin{cases} x=0 & V=V_1 \\ x=a & V=V_2 \\ y=0,b & V=0 \end{cases}$$

$c_2 = -k^2$

$$\frac{1}{Z}\frac{d^2 Z}{dz^2} = C_3 = 0$$

$$\frac{d^2 Y}{dy^2} = c_2 Y$$

$$\begin{cases} Y = A\sin(ky) + B\cos ky & c_2 < 0 \\ Y = A' e^{\sqrt{c_2}\, y} + B' e^{-\sqrt{c_2}\, y} & c_2 > 0 \\ Y(y) = A'' + B'' y & c_2 = 0 \end{cases}$$



$$Y(y) = A\sin ky + B\cos ky$$

$$Y(0) = 0 = A\underbrace{\sin(0)}_{0} + B\underbrace{\cos(0)}_{1} = B = 0$$

$$Y(y) = A\sin ky$$

$$Y(b) = A\sin(kb) = 0 \qquad kb = n\pi$$

$n = 1, 2, 3 \ldots$

$$k = \frac{n\pi}{b}$$

$$\underset{-k^2}{C_1 + C_2} + \underset{0}{C_3} = 0 \qquad \boxed{C_1 = k^2}$$