Free Fall - Linear Friction

Suppose we have a mass, m, which is permitted to free-fall in the atmosphere near the average-surface of the Earth. Assume that the dissipative force of friction is assumed to be proportional to the velocity, v, of the mass and from Newton's laws derive a differential equation modeling the velocity of the mass as a function of time. Solve this differential equation assuming the following initial conditions and describe the long-term asymptotic of the dynamics of the mass,

1. $y(0) < \dfrac{mg}{\gamma}$,
2. $y(0) = \dfrac{mg}{\gamma}$,
3. $y(0) > \dfrac{mg}{\gamma}$,

where g is the gravitational constant of the universe and γ is the coefficient of kinetic friction. Assuming that $m = g = \gamma = 1$ solutions to various initial value problems and comment on the results.

Free Fall - Quadratic Friction

It has been argued that for certain geometries dissipative forces are proportional to the square of the velocity. You might want to think about this in the same sense that Taylor's theorem argues that if a curve is locally concave then a quadratic function would give a better local approximation than a linear function. That is, the relation between force and velocity need not be as simple as a linear one. Under these new assumptions derive from Newton's laws a mathematical model that models the velocity of the mass as a function of time. Solve this differential equation assuming the following initial conditions and describe the long-term asymptotic of the dynamics of the mass,

1. $y(0) < \dfrac{mg}{\gamma}$,
2. $y(0) = \dfrac{mg}{\gamma}$,
3. $y(0) > \dfrac{mg}{\gamma}$,

where g is the gravitational constant of the universe and γ is the coefficient of kinetic friction. Assuming that $m = g = \gamma = 1$ solutions to various initial value problems and comment on the results.

Hyperbolic Trigonometric Functions

We will see, later in this class, that the exponential function is fundamentally related to the sine and cosine functions. This can be seen through the use of Taylor's theorem and the complex number system. It is because of this relationship that exponential function can be made to behave similarly to trigonometric functions in terms of differential and integral calculus. These functions casually come up in practice to compactly write solutions to common physical problems. In response to this I have constructed the following list of definitions and properties, which may or may not be useful.

We begin with the following definitions for hyperbolic cosine and hyperbolic sine:[1]

$$\cosh(x) = \frac{1}{2}\left(e^{x}+e^{-x}\right) \tag{1}$$
$$\sinh(x) = \frac{1}{2}\left(e^{x}-e^{-x}\right) \tag{2}$$

From (1)-(2) the following can be derived:

1. Symmetry Properties:

$$\sinh(-x) = -\sinh(x) \tag{3}$$
$$\cosh(-x) = \cosh(x) \tag{4}$$

2. Definition of Hyperbolic Tangent:

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} \tag{5}$$

3. Some Hyperbolic Trigonometric Identities:[2]

$$(\cosh(x))^2 - (\sinh(x))^2 = 1 \tag{6}$$
$$(\operatorname{sech}(x))^2 + (\tanh(x))^2 = 1 \tag{7}$$

4. Rules of Differentiation:

$$\frac{d\left[\cosh(x)\right]}{dx} = \sinh(x) \tag{8}$$
$$\frac{d\left[\sinh(x)\right]}{dx} = \cosh(x) \tag{9}$$
$$\frac{d\left[\tanh(x)\right]}{dx} = (\operatorname{sech}(x))^2 \tag{10}$$

5. Standard Integrals:

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \operatorname{arcsinh}\left(\frac{x}{a}\right) + C \tag{11}$$
$$\int \frac{dx}{\sqrt{x^2-a^2}} = \operatorname{arccosh}\left(\frac{x}{a}\right) + C \tag{12}$$
$$\int \frac{dx}{a^2-x^2} = \frac{1}{a}\operatorname{arctanh}\left(\frac{x}{a}\right) + C, \quad x^2 < a^2 \tag{13}$$

[1] Wikipedia, `http://en.wikipedia.org/wiki/Hyperbolic_function`, has the following useful overview of these functions: Just as the points $(x, y) = (\cos t, \sin t)$, $t \in \mathbb{R}$ form a circle with a unit radius, the points $(x, y) = (\cosh t, \sinh t)$, $t \in \mathbb{R}$ form the right half of the equilateral hyperbola. Hyperbolic functions are also useful because they occur in the solutions of some important linear differential equations, notably that defining the shape of a hanging cable, the catenary, and Laplace's equation (in Cartesian coordinates), which is important in many areas of physics including electromagnetic theory, heat transfer, fluid dynamics, and special relativity.

[2] A more complete listing may be found here, `http://www.alcyone.com/max/reference/maths/hyperbolic.html`