

$$\vec{E}_0 = E_0 e^{i\omega t} \hat{z}$$

$$\vec{E}_0 \text{ is time varying}$$

$$\vec{E}_0 \rightarrow \vec{B}_1 = \frac{\mu_0 I}{2\pi r^2} s \hat{\phi} = i \frac{E_0}{2c} (\frac{\omega s}{c}) \hat{\phi}$$

Ampere Maxwell Law

$$\vec{B}_1 \text{ is time varying}$$

$$\vec{B}_1 \rightarrow \vec{E}_2 \quad (\text{iterative approach})$$

From Faraday's Law

$$\oint \vec{E} \cdot d\vec{x} = - \frac{d}{dt} \int \vec{B} \cdot d\vec{a}$$

$$\oint \vec{E}_0 \cdot d\vec{x} + \oint (\vec{E}_0 + \vec{E}_2) \cdot d\vec{x} = - \int \frac{\partial \vec{B}_1}{\partial t} \cdot d\vec{a}$$

$$\oint \vec{E}_0 \cdot d\vec{x} = 0 \quad \text{is constant in space}$$

$$\oint \vec{E}_2 \cdot d\vec{x} = E_2(s) h - E_2(0) h + 0 + 0$$

$$E_2(0) \equiv 0$$

$$\Rightarrow \oint \vec{E} \cdot d\vec{x} = h E_2(s) = - \int \frac{\partial \vec{B}_1}{\partial t} \cdot d\vec{a}$$

$$h E_2(s) = \int_0^s \frac{\partial}{\partial t} i E_0 e^{i\omega t} \frac{\omega s}{2c^2} \times h dx$$

$$= \int_0^s \frac{i \omega^2}{2c^2} E_0 e^{i\omega t} h x dx$$

$$h E_2(s) = - \frac{\omega^2}{2c^2} E_0 e^{i\omega t} h \int_0^s x dx$$

$$E_2(s) = - \frac{\omega^2}{4c^2} E_0 e^{i\omega t} s^2$$

$$\vec{E}_2 = - \frac{E_0 e^{i\omega t}}{4} \left(\frac{\omega s}{c} \right)^2 \hat{z}$$

$$\vec{E}_2 \rightarrow \vec{B}_3 \quad \text{from Ampere's Law}$$

$$\oint \vec{B} \cdot d\vec{x} = \frac{1}{c^2} \frac{\partial}{\partial t} \int \vec{E} \cdot d\vec{a}$$

$$\oint (\vec{B}_1 + \vec{B}_3) \cdot d\vec{x} = \frac{1}{c^2} \frac{\partial}{\partial t} \int (\vec{E}_0 + \vec{E}_2) \cdot d\vec{a}$$

$$\oint \vec{B}_3 \cdot d\vec{x} = \frac{1}{c^2} \frac{\partial}{\partial t} \int \vec{E}_2 \cdot d\vec{a}$$

$$\vec{E} = \vec{E}_0 + \vec{E}_2 + \vec{E}_4 + \dots$$

$$= E_0 e^{i\omega t} \left[1 - \frac{1}{4} \left(\frac{s\omega}{c} \right)^2 + \frac{\sqrt{10}}{16} \left(\frac{s\omega}{c} \right)^4 \dots \right] \hat{z}$$

$$= E_0 e^{i\omega t} J_0 \left(\frac{s\omega}{c} \right) \hat{z} \quad ?$$

Continuity Equation



Volume V with boundary S

$$Q = \int_V \rho d\tau \quad \text{total charge in } V$$

$$\text{Current flowing out of } V = \oint_S \vec{J} \cdot d\vec{a}$$

Local charge conservation would give

$$\frac{dQ}{dt} = - \oint_S \vec{J} \cdot d\vec{a} \quad \xrightarrow{\text{divergence Thm}}$$

$$\frac{d}{dt} \left(\int_V \rho d\tau \right) = - \int_V (\vec{\nabla} \cdot \vec{J}) d\tau$$

$$\int_V \frac{\partial \rho}{\partial t} d\tau = - \int_V (\vec{\nabla} \cdot \vec{J}) d\tau$$

Nothing special about V

$$\Rightarrow \frac{\partial \rho}{\partial t} = - \vec{\nabla} \cdot \vec{J} \quad \text{Continuity Eqn}$$

This is actually built into Maxwell's Equations:

$$\text{Ampere's Law: } \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Take divergence

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = \mu_0 \vec{\nabla} \cdot \vec{J} + \mu_0 \epsilon_0 \vec{\nabla} \cdot \frac{\partial \vec{E}}{\partial t}$$

$$\text{Zero for my vector field } 0 = \mu_0 \vec{\nabla} \cdot \vec{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} (\vec{E} \cdot \vec{E})$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad (\text{Gauss's Law})$$

$$0 = \mu_0 \vec{\nabla} \cdot \vec{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(\frac{\rho}{\epsilon_0} \right)$$

$$0 = \vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} \quad \text{or} \quad \frac{\partial \rho}{\partial t} = - \vec{\nabla} \cdot \vec{J}$$

Poynting Theorem (Work-Energy Thm)

$$U_{\text{mech}} = \int u_{\text{mech}} dz$$

volume landed by

$$\frac{dU_{\text{mech}}}{dt} = \int \frac{\partial u_{\text{mech}}}{\partial t} dz$$

Now we have fields $\vec{E} \neq \vec{B}$

$$W_{dq} = \vec{F} \cdot d\vec{x} \quad \text{in some time } dt$$

$\frac{d\vec{x}}{dt} = \vec{v} \Rightarrow d\vec{x} = \vec{v} dt$

$$W_{dq} = \vec{F} \cdot \vec{v} dt \quad \text{Lorentz force Law}$$

$$= dq (\vec{E} + \vec{v} \times \vec{B}) \cdot \vec{v} dt$$

$$\vec{F} = (\vec{v} \times \vec{B}) \cdot \vec{v} = 0 \quad (\text{Magnetic fields don't do work})$$

$$W_{dq} = dq \vec{E} \cdot \vec{v} dt$$

$$dq = \rho d\tau$$

$$W_{dq} = dt d\tau \vec{E} \cdot \vec{v} \rightarrow \vec{J}$$

$$= dt d\tau \vec{E} \cdot \vec{J}$$

To find total work done in the volume in time dt

$$dW = \int w_{dq} = \int dt dz \vec{E} \cdot \vec{J}$$

$$= dt \int_v \vec{E} \cdot \vec{J} dz$$

$$\frac{dW}{dt} = \int_v \vec{E} \cdot \vec{J} dz \quad \text{Volume Power density}$$

This Power will lead to increase U_{mech}

Work-Energy Thm:

$$\frac{dW}{dt} = \frac{dU_{\text{mech}}}{dt}$$

$$\int_v \vec{E} \cdot \vec{J} dz = \int_v \frac{\partial u_{\text{mech}}}{\partial t} dz$$

$$\frac{\partial u_{\text{mech}}}{\partial t} = \vec{E} \cdot \vec{J} \rightarrow \text{get rid of } \vec{J} \text{ in terms of the fields using}$$

$$\frac{\partial u_{\text{mech}}}{\partial t} = \vec{E} \left[\mu_0 \vec{J} \times \vec{B} - \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right] \quad \text{Maxwell's Eqns}$$

$$= \frac{\mu_0}{\mu_0} \vec{E} \cdot (\vec{J} \times \vec{B}) - \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$= \frac{\mu_0}{\mu_0} \vec{E} \cdot (\vec{J} \times \vec{B}) - \frac{1}{2} \epsilon_0 \frac{\partial}{\partial t} (\vec{E}^2)$$

$$\vec{E} \cdot (\vec{J} \times \vec{B}) = \vec{B} \cdot (\vec{J} \times \vec{E}) - \vec{E} \cdot (\vec{J} \times \vec{B})$$

Product rule 6
 $\vec{E} \cdot (\vec{J} \times \vec{B}) = (\vec{B} \cdot (\vec{J} \times \vec{E})) - (\vec{B} \cdot (\vec{J} \times \vec{B}))$

$$\frac{\partial u_{\text{mech}}}{\partial t} = \frac{1}{\mu_0} \left[\vec{B} \cdot (\vec{J} \times \vec{E}) - \vec{B} \cdot (\vec{J} \times \vec{B}) \right] - \frac{1}{2} \epsilon_0 \frac{\partial}{\partial t} (\vec{E}^2)$$

$$\frac{\partial \underline{U}_{\text{mech}}}{\partial t} = \frac{1}{\mu_0} \left[\vec{B} \cdot (\vec{\nabla} \times \vec{E}) - \vec{\nabla} \cdot (\vec{E} \times \vec{B}) \right] - \frac{1}{2} \epsilon_0 \frac{\partial}{\partial t} (E^2)$$

$$= \frac{1}{\mu_0} \left[-\frac{1}{2} \frac{\partial}{\partial t} (\vec{B}^2) - \vec{\nabla} \cdot (\vec{E} \times \vec{B}) \right] - \frac{1}{2} \epsilon_0 \frac{\partial}{\partial t} (E^2)$$

$$\frac{\partial \underline{U}_{\text{mech}}}{\partial t} + \frac{1}{2} \epsilon_0 \frac{\partial}{\partial t} (E^2) + \frac{1}{2\mu_0} \frac{\partial}{\partial t} (\vec{B}^2) = -\frac{1}{\mu_0} \vec{\nabla} \cdot (\vec{E} \times \vec{B})$$

$$U_{\text{em}} = \frac{1}{2} \epsilon_0 E^2 + \frac{1}{2\mu_0} \vec{B}^2$$

$$\frac{\partial}{\partial t} (U_{\text{mech}} + U_{\text{em}}) = -\vec{\nabla} \cdot \vec{s} \quad \vec{s} = \frac{1}{\mu_0} (\vec{E} \times \vec{B})$$

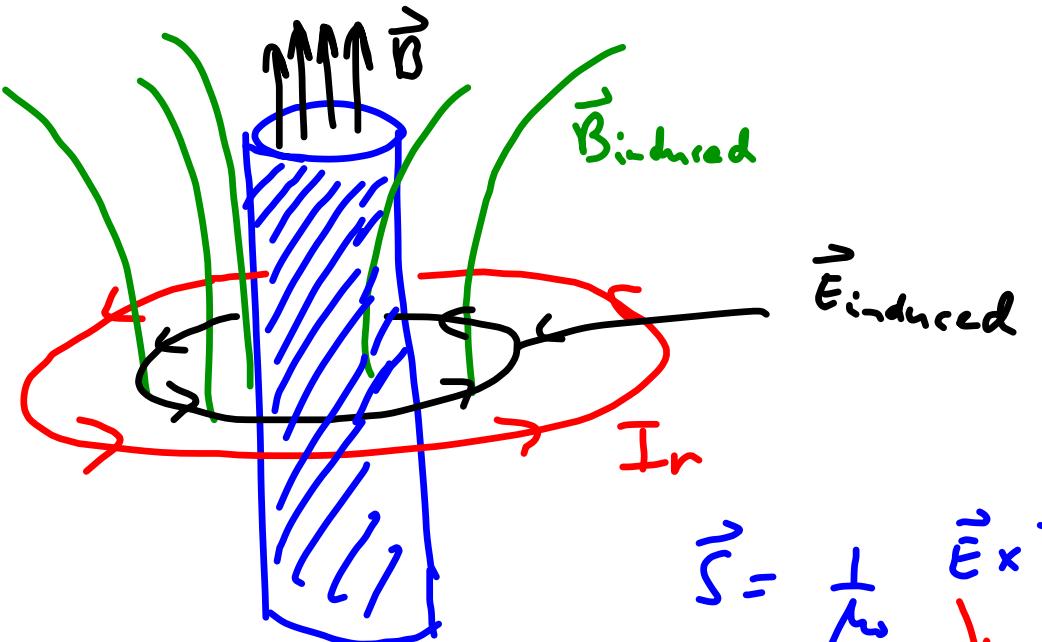
Compare to

$$\frac{\partial}{\partial t} \rho = -\vec{\nabla} \cdot \vec{j}$$

$$\begin{aligned} \frac{d}{dt} U_{\text{mech}} + \frac{d}{dt} U_{\text{em}} &= - \int_v (\vec{\nabla} \cdot \vec{s}) d\tau \\ &= - \oint \vec{s} \cdot d\vec{a} \end{aligned}$$

just like

$$\frac{d\vec{a}}{dt} = - \oint \vec{j} \cdot d\vec{a}$$



$$\vec{s} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

\vec{E}_{induced} \vec{B}_{induced}