Radiation: relativistic approach (14.9)

slow-moving charge $P = \frac{2}{3}\frac{e^2 a^2}{c^3}$

this is exact in a ref. frame instantaneously at rest w/ charge.
- accelerating ref. frame is not a Lorentz frame.

$\therefore$ at an instant t, calc rad. from charge in a ref frame K' moving w/ charge.
- then transform back to observer in K

4-vector vel: $U = (\vec{u}, ic)(1 - u^2/c^2)^{-1/2}$

4-vector accel: $\mathbb{D} = dU/d\tau$

remember $d\tau \equiv \sqrt{dt^2 - \frac{dx_i dx_i}{c^2}} = dt\sqrt{1 - u^2/c^2}$

so
$$\mathbb{D} = \frac{1}{\sqrt{1-\beta^2}}\frac{d}{dt}\left((1 - u^2/c^2)^{-1/2}(\vec{u}, ic)\right)$$
$$= \frac{1}{\sqrt{\quad}}\left[\frac{\vec{u}\cdot\dot{\vec{u}}/c^2}{(\quad)^{3/2}}(\vec{u}, ic) + (\quad)^{-\frac{1}{2}}(\dot{\vec{u}}, 0)\right]$$
$$= \left(\frac{\dot{\vec{u}}}{1-\beta^2} + \frac{\vec{u}(\vec{u}\cdot\dot{\vec{u}})}{c^2(1-\beta^2)^2}, \frac{i(\vec{u}\cdot\dot{\vec{u}})}{c(1-\beta^2)^2}\right)$$

this is good for a particle moving at $\vec{u}$ in the ref. frame.
in K' (moving with charge) $\vec{u}' = 0$

so $\mathbb{D}' = (\dot{\vec{u}}', 0)$ as expected.

Now transform back to K:

$\mathbb{D}_\nu = \lambda_{\mu\nu}\mathbb{D}'_\mu$ assuming $\vec{u} = u\hat{z}$ and $t = 0$

$$\begin{pmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{pmatrix} = \begin{pmatrix} \dot{u}_1' \\ \dot{u}_2' \\ D_3'\gamma \\ D_3'(i\beta\gamma) \end{pmatrix} = \begin{pmatrix} \dot{u}_1' \\ \dot{u}_2' \\ \dot{u}_3'/\sqrt{1-\beta^2} \\ i\beta\dot{u}_3'/\sqrt{1-\beta^2} \end{pmatrix}$$

note that even though $\vec{u}$ is along $\hat{z}$, $\dot{\vec{u}}$ may not be.
also $\beta = u/c$ w/ u = mag of vel.

Still need to connect this result w/ accel. compon in lab frame:

recall $\vec{u} = u_3\hat{z} = (0,0,u)$; $\gamma = (1-\beta^2)^{-1/2}$

$D_1 = \dot{u}_1\gamma^2 = \dot{u}_1'$

$\rightarrow$ $\underline{\dot{u}_1 = \dot{u}_1'(1-\beta^2)}$

and $\underline{\dot{u}_2 = \dot{u}_2'(1-\beta^2)}$

$$D_3 = \dot{u}_3\gamma^2 + \frac{u_3(u_3\dot{u}_3)}{c^2}\gamma^4 = \dot{u}_3'\gamma$$

$$= \dot{u}_3\gamma^2\left(1 + \frac{u^2}{c^2}\gamma^2\right) = \dot{u}_3\gamma^2\left(1 + \frac{\beta^2}{1-\beta^2}\right) = \dot{u}_3\gamma^2\left(\frac{1}{1-\beta^2}\right)$$

$\dot{u}_3\gamma^4 = \dot{u}_3'\gamma$ or $\underline{\dot{u}_3 = \dot{u}_3'(1-\beta^2)^{3/2}}$

now we can represent accel $(a')^2$ in terms of accel components in rest frame K

$$a'^2 = \sum \dot{u}_i'\dot{u}_i' = \frac{\dot{u}_1^2 + \dot{u}_2^2}{(1-\beta^2)^2} + \frac{\dot{u}_3^2}{(1-\beta^2)^3}$$

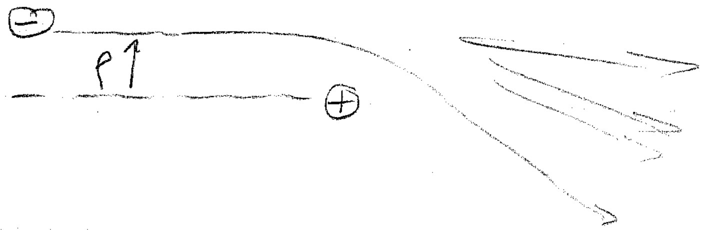
general direction for $\vec{u}$ is in book.

$$P = P' = \frac{2}{3}\frac{e^2 a'^2}{c^3}$$

$\vec{u} \parallel \dot{\vec{u}}$ $\quad P = \frac{2}{3}\frac{e^2 a^2}{c^3}\gamma^6$ $\quad$ linear accel.

$\vec{u} \perp \dot{\vec{u}}$ $\quad P = \frac{2}{3}\frac{e^2 a^2}{c^3}\gamma^4$ $\quad$ cyclotron.

Bremsstrahlung radiation: (non relativistic)
"braking" radiation



emitted power

$$P = \frac{2}{3}\frac{e^2 a^2}{c^3}$$

total emitted:

$$\Delta E = \int_{-\infty}^{\infty} P(t)\,dt = \frac{2}{3}\frac{e^2}{c^3}\int_{-\infty}^{\infty}(a(t))^2\,dt$$

we want the spectrum

$\therefore$ write $a(\nu) = \int_{-\infty}^{\infty} a(t)\, e^{-i 2\pi\nu t}\,dt$

by Parseval's theorem $\int_{-\infty}^{\infty}|f(t)|^2\,dt = \int_{-\infty}^{\infty}|F(\nu)|^2\,d\nu$

since $a(t)$ is real
$a(\nu)$ is even

$$\therefore \int_{-\infty}^{\infty}|a(\nu)|^2\,d\nu = 2\int_0^{\infty}|a(\nu)|^2\,d\nu$$

$$\rightarrow \Delta E = \frac{4}{3}\frac{e^2}{c^3}\int_0^{\infty}|a(\nu)|^2\,d\nu = \int_0^{\infty} P_\nu\,d\nu$$

$P_\nu\,d\nu$ = power radiated btw ν and $\nu + d\nu$
= spectral power density.

Now assume classical trajectory is nominally unchanged

$\rightarrow$ well-defined path, $a(t)$ which is $f(v_0, \rho)$

V_0 = init. veloc.

ρ = impact parameter.

Suppose we have a beam of electrons with flux

$$N_e V = \#/s/cm^2$$

each emits $\Delta E(v, \rho)$

radiated power is (per ion)

$$q = N_e v \int_0^\infty \Delta E(v,\rho)\, 2\pi\rho\, d\rho$$

can represent this in terms of the spectrum $\int_0^\infty dq_\nu = q$

$$dq_\nu = N_e v\, d\nu \int_0^\infty P_\nu\, 2\pi\rho\, d\rho$$

in a plasma,

the electrons have a velocity distribution

$f(v)\,dv$ = prob. an electron has vel. v

ion density N_i

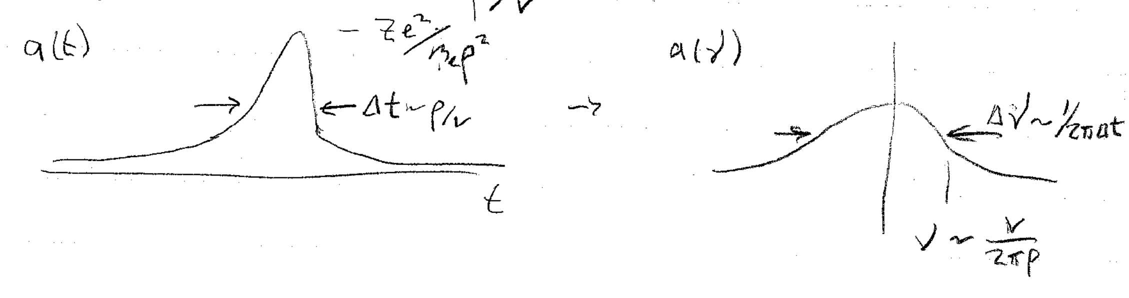
$$N_i\, dq_\nu = N_i N_e\, d\nu \int_0^\infty dv\, f(v)\, v \int_0^\infty P_\nu(v,\rho)\, 2\pi\rho\, d\rho$$

estimate integral over ρ:

$$\text{force} = -\frac{Ze^2\vec{r}}{r^3} \rightarrow \text{accel} = -\frac{Ze^2\vec{r}}{m_e r^3}$$

max accel is approx $r \sim \rho$

acts during time $\Delta t \sim \rho/v$



∴ connection btw emitted frequencies + impact param

estimate: emitted freq. near ν

come from elect w/ impact param $\rho \sim v/2\pi\nu$

electrons btw ρ and $\rho + d\rho$ → rad btw ν and $\nu + d\nu$

$$d\rho \sim \frac{v}{2\pi\nu^2}\, d\nu \sim \frac{v}{2\pi}\left(\frac{2\pi\rho}{v}\right)^2 d\nu = \frac{2\pi\rho^2}{v}\, d\nu$$

estimate rad. energy given off by electron at ρ

$$\Delta E \sim \frac{2}{3}\frac{e^2}{c^3} a^2 \Delta t \sim \frac{2}{3}\frac{e^2}{c^3}\left(\frac{Ze^2}{m_e\rho^2}\right)^2 \frac{\rho}{v}$$

$$\sim \frac{2Z^2e^6}{3c^3 m_e^2 \rho^3 v}$$

extend this to spectrum:

$$dq_\nu \sim \Delta E\, 2\pi\rho\, d\rho \sim \frac{4\pi}{3}\frac{Z^2e^6\, d\rho}{c^3 m_e^2 \rho^2 v} \sim \frac{8\pi^2}{3}\frac{Z^2e^6}{m_e^2 c^3 v^2}\, d\nu$$

exact calc:

for $\nu \gg \frac{mv^3}{2\pi Z e^2}$ or $\frac{mv^2}{Ze^2/\rho} \ll 2\pi\nu\Delta t$ (high freq).

$$dq_\nu = \frac{32\pi^2}{3\sqrt{3}}\frac{Z^2e^6}{m^2c^3v^2}\, d\nu$$ only 2.3 x larger.

low freq: $\nu \ll mv^3/2\pi Z e^2$

$$dq_\nu = \frac{32\pi^2}{3}\frac{Z^2e^6}{m^2c^3v^2}\ln\left(\frac{mv^3}{1.78\pi\nu Z e^2}\right) d\nu$$

estimates not so good for lowest freq.

- low freq. come from weak scatter at large ρ.
- these are usually cut off by Debye length.

small ρ → quantum corrections, cutoff from Debroglie λ

→ upper limit to ν is at $h\nu = \frac{1}{2}mv^2$ if free-free.

- elect. can also combine → free-bound, recomb. emission.