Lecture 18: Invariant quantities + Minkowski formulation of relativity

Last time we discussed the notion of covariance: That physical laws retain their form as you go from one frame to another. So if $\nabla \times \vec{E} = -\partial \vec{B}/\partial t$, we expect that $\nabla' \times \vec{E}' = -\partial \vec{B}'/\partial t'$

We also spoke of invariant quantities, like q. We assume q is the same in every frame, so $q = q'$.

In any theory, we want to write as many covariant laws as possible, with as many invariant quantities as possible.

We can draw a really productive analogy to a piece of classical theory: The <u>length</u> of vectors in invariant under <u>rotation</u>.

Consider a position vector: $\vec{x} = x\hat{i} + y\hat{j} + z\hat{k}$

And consider a rotation through angle θ about the z-axis. We represent this in matrix form via:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \\ z \end{bmatrix}$$

$\vec{x}$ is clearly not invariant under rotation. But its length, which we obtain with a dot product, is:

$$\vec{x} \cdot \vec{x} = x^2 + y^2 + z^2$$

$$\vec{x}' \cdot \vec{x}' = x^2\cos^2\theta + y^2\sin^2\theta - 2xy\cos\theta\sin\theta + x^2\sin^2\theta + y^2\cos^2\theta + 2xy\cos\theta\sin\theta + z^2$$

$$= x^2 + y^2 + z^2 = \vec{x} \cdot \vec{x}$$

The fact that vectors have lengths that are the same no matter your point of view is kind of a thing we use a lot. It's even the case that spatial intervals $(\vec{x}_1 - \vec{x}_2)$ are invariant under Galilean transforms, too. "Length" is useful because it is so reliably invariant.

But spatial intervals aren't Lorentz invariant, in general. $\Delta\vec{x} \cdot \Delta\vec{x} \neq \Delta\vec{x}' \cdot \Delta\vec{x}'$ That's what Lorentz contraction expresses, when you get down to it.

And in any case, we're supposed to be describing events in spacetime, with x, y, z, and also t. The x + t coordinates can't be disentangled, as we see in the Lorentz transforms.

So to specify an event in spacetime, we need to write a vector with four elements:

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$ We throw a factor of c to match units.
(this does one other special thing, which I'll mention shortly)

Now, Lorentz transforms are linear, so we can write a transform in matrix form:

$$\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(t - \frac{vx}{c^2}\right) \end{aligned}$$ becomes $$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

This is really reminiscent of rotations, for which there was a thing called length that was invariant. Do 4-vectors have a "length" that is invariant under Lorentz transforms? It turns out they do.

If $x^2+y^2+z^2$ is the (squared) length of a 3-vector, let $x^2+y^2+z^2-(ct)^2$ be the (squared) length of a 4-vector. Let's check the invariance explicitly

$$\begin{aligned} ct' &= \gamma ct - \beta\gamma x \\ x' &= -\beta\gamma ct + \gamma x \\ y' &= y \\ z' &= z \end{aligned}$$

$$\Rightarrow x'^2 + y'^2 + z'^2 - (ct')^2 = (\gamma x - \beta\gamma ct)^2 + y^2 + z^2 - (\gamma ct - \beta\gamma x)^2$$
$$\gamma^2 x^2 + \beta^2\gamma^2 c^2 t^2 + y^2 + z^2 - \gamma^2 c^2 t^2 - \beta^2 \gamma^2 x^2$$
$$= x^2\gamma^2(1-\beta^2) + y^2 + z^2 - c^2t^2\gamma^2(1-\beta^2)$$

And since $\gamma^2 = \frac{1}{1 - v^2/c^2} = \frac{1}{1-\beta^2}$, $\gamma^2(1-\beta^2) = 1$, so
$$x'^2 + y'^2 + z'^2 - (ct')^2 = x^2 + y^2 + z^2 - (ct)^2$$

Also note that this definition of length retains its invariance under rotations, since rotations affect x, y, z only. So this really represents an <u>extension</u> of, not a replacement of, the classical notion of length.

Here's some notation. Classically, we accessed length via a dot product,
$$\vec{x}\cdot\vec{x} = x_i x_i = x_1^2 + x_2^2 + x_3^2 = x^2 + y^2 + z^2$$

Note that $x_i x_i$ is summation notation for $\sum_{i=1}^{3} x_i x_i$ (implied summation when indexes repeat)

We need a dot product generalized to spacetime (4D). And we need to rig it so that the $(ct)^2$ part of it is negative. We do it like so:

Let x_μ represent the μ^{th} component of (ct, x, y, z)

The naive 4D dot product would be $x_\mu x_\mu = (ct)^2 + x^2 + y^2 + z^2$

Instead, define the Minkowski product as

(not squared/cubed... those are indices)

$$x^\mu x_\mu = x^0 x_0 + x^1 x_1 + x^2 x_2 + x^3 x_3$$

$$= -x_0 x_0 + x_1 x_1 + x_2 x_2 + x_3 x_3$$

Bookkeeping rule: With 4-vectors, write dot products using one subscripted index and one <u>superscripted</u> index. Moving an index up or down doesn't do anything if $\mu = 1,2,3$, but moving the 0 index incurs a minus sign.

Don't overthink it. $-x_0 x_0 + x_1 x_1 + x_2 x_2 + x_3 x_3$ is the quantity invariant under both rotations and Lorentz transforms. So we extended our definition of dot product and made a notation to help keep track. That's it.

A bit more vocab:

x_μ : covariant vector (different sense of the word)
x^μ : contravariant vector

x_i, i is any Roman letter: Implies index that runs from 1 to 3

x_μ, μ is any Greek letter: Implies index that runs from 0 to 3

Ok, so the Minkowski formulation of special relativity basically involves writing every quantity as part of a 4-vector (a vector whose length is Lorentz invariant), and writing every physical law in a manifestly covariant form. Then all the relativistic stuff kind of happens automatically.

So position x_μ is a four-vector. What is the four velocity?
Classically, $v_i = dx_i/dt$

If we write $v_\mu = dx_\mu/dt = \begin{pmatrix} c \\ dx/dt \\ dy/dt \\ dz/dt \end{pmatrix}$

that is a vector with four components, and is a velocity, but it's not a 4-vector because its length is certainly <u>not</u> Lorentz invariant. Regular speeds change quite a bit as you go from frame to frame... which is kind of obvious when you think about it.

So we want to come up with a way to express velocity such that the "length" is Lorentz invariant. Here's an idea:

Define τ to be the proper time, the time as measured by an object in its rest frame. This <u>is</u> Lorentz invariant... If Steve, Carl, and Hank are all in different frames, their watches disagree, but they can all at least agree on <u>what</u> <u>Hank's</u> <u>watch</u> <u>says</u>.

So intervals of proper time $d\tau$ are Lorentz invariant. And x_μ is a 4-vector, so dx_μ is, too

Thus $\dfrac{dx_\mu}{d\tau} \equiv \eta_\mu$ represents a 4-vector, the true 4-velocity.

Note that $d\tau = dt/\gamma$ (time dilation, basically) so

$$\eta_\mu = \begin{pmatrix} \gamma c \\ \gamma v_x \\ \gamma v_y \\ \gamma v_z \end{pmatrix}$$

Note that the components of η_μ <u>do</u> change as you go from frame to frame. All observers are using some agreed-upon clock, but they're measuring spatial intervals differently. It's just that the components change in offsetting ways so that $\eta_\mu \eta^\mu = \eta'_\mu \eta'^\mu$.

Let's do momentum next. Classically, $\bar{p} = m\bar{v}$. If we posit that the (rest) mass of an object is Lorentz invariant, then given that η_μ is a 4-vector, so would be $p_\mu = m\eta_\mu$.
This is the 4-momentum: $p_\mu = \begin{pmatrix} \gamma mc \\ \gamma m v_x \\ \gamma m v_y \\ \gamma m v_z \end{pmatrix}$

And hey, classically, $\bar{F} = d\bar{p}/dt$, which isn't covariant. But

$\dfrac{dp_\mu}{d\tau} \equiv K_\mu$ <u>would</u> be. K_μ is the <u>Minkowski</u> <u>force</u>, and

$K_\mu = dp_\mu/d\tau$ is what we call the manifestly (fancy word for obviously) covariant form of Newton's 2nd.

But wait... what's the 0^{th} component of p_μ + K_μ?
Well, back in modern, you deduced that γmc^2 is the total (rest + kinetic) energy of an object. We won't repeat the derivation here. But we note that $p_0 = \gamma mc = E/c$.

So $p_\mu = \begin{pmatrix} E/c \\ m\eta_x \\ m\eta_y \\ m\eta_z \end{pmatrix}$ is sometimes called the energy-momentum 4-vector.

And $K_\mu = \begin{pmatrix} \frac{1}{c}\, dE/d\tau \\ \gamma F_x \\ \gamma F_y \\ \gamma F_z \end{pmatrix}$ F: the classical force.

The components of p_μ describe the energy & momentum of an object, and the components of K_μ describe the rates of change of the energy & momenta (power & forces on it).

Note that $-E^2/c^2 + m(\eta_x^2 + \eta_y^2 + \eta_z^2)$ is an invariant quantity, so if it were conserved in any frame, it'd be conserved in every frame.