Sample questions

Note Title 10/5/2007

(1) A Hermitian matrix satisfies

$$A^+ = A$$

An anti-Hermitian matrix satisfies $A^+ = -A$

Show that any matrix can be written as the sum of a Hermitian and anti-Hermitian matrix

$$A = \frac{A + A^+}{2} + \frac{A - A^+}{2}$$ identity

$$\left(\frac{A + A^+}{2}\right)^+ = \frac{A^+ + (A^+)^+}{2}$$

$$= \frac{A^+ + A}{2}$$ Hermitian

$$\left(\frac{A - A^+}{2}\right)^+ = \frac{A^+ - A}{2} = -\left(\frac{A - A^+}{2}\right)$$

Anti Hermit.

(2) Suppose

$$g(x) = \sum_{n=0}^{\infty} a_n x^n \qquad (1)$$

$$f(x) = \sum_{n=0}^{\infty} b_n x^n \qquad (2)$$

Show that $g(x) f(x) =$

$$\sum_{n=0}^{\infty} \underline{\left(\sum_{i=0}^{n} a_i b_{n-i} \right)} x^n$$

careful: remember that the indices in (1) + (2) are dummy summation indices.

The underlined expression above is the "convolution" of the 2 sequences $\{a_n\}$ and $\{b_n\}$

$$g(x) f(x) = \sum_{n=0}^{\infty} a_n x^n \sum_{m=0}^{\infty} b_m x^m$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n b_m x^{(n+m)}$$

$n + m = i$

$m = i - n$

$$\underbrace{\sum_{i=0}^{\infty} \sum_{n=0}^{i} a_n b_{i-n}}_{\text{called the convolution}} x^i$$

$\equiv a \otimes b$

$$\sum_{i=0}^{\infty} [a \otimes b]_i \, x^i = g(x) f(x)$$

(3) compute the first 3 nonzero terms in the McClaurin series for

i) $\frac{1}{1-x}$ $1+x+x^2$

ii) $\sin(x)$ $x-\frac{x^3}{6}+\frac{x^5}{120}$

iii) $\ln(1+x)$ $x-\frac{x^2}{2}+\frac{x^3}{3}$

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(4) compute the first two terms of the McClaurin series for $\sin(x)$, $\cos(x)$.

use these to derive the following approximation for $\cot(x)$

$$\cot(x) \approx \frac{1}{x}-\frac{x}{3} \quad \text{for } |x| \ll 1$$

given that $\cot(.5) = 1.83049$ how big is the error of this approx at $x=.5$? Estimate this. You don't need a calculator

$$\frac{\cos(x)}{\sin(x)} \approx \frac{1 - x^2/2}{x - x^3/6}$$

Note: for small enough x

$\left.\begin{array}{l}\cos x \simeq 1 \\ \sin x \approx x\end{array}\right\}$ so we should

expect $\cot x$ to go like

$\frac{1}{x} + \cdots$ so write

$$\frac{1 - x^2/2}{x - x^3/6} = \frac{1}{x} + f(x)$$

$$= \frac{3}{x}\left(\frac{2 - x^2}{6 - x^2}\right) = \frac{1}{x} + f(x)$$

$$\Rightarrow f(x) = \frac{1}{x}\left[3\,\frac{2 - x^2}{6 - x^2} - 1\right]$$

$$= \frac{1}{x}\left[\frac{6 - 3x^2 - (6 - x^2)}{6 - x^2}\right]$$

$$= \frac{1}{x}\left[\frac{-2x^2}{6 - x^2}\right]$$

$$= \frac{1}{x}\left[\frac{2}{1 - 6/x^2}\right]$$

for small x $\rightarrow$ $\frac{1}{x}\left[-\frac{x^2}{3}\right] = \underbrace{-\frac{x}{3}}_{f(x)}$

so

$$\boxed{\cot x \approx \frac{1}{x} - \frac{x}{3}}$$

mathematica says: $\cot(x) = \frac{1}{x} - \frac{x}{3} + \cdots$ ✓

5) $\int \frac{1}{1+x^2} = \text{Arctan}(x)$

compute the McClaurin series for $\frac{1}{1+x^2}$ and use your knowledge of the tangent function to show that

$$\pi = 4\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots\right]$$

i.e. $\pi = 4\sum_{N=1}^{\infty} \frac{(-1)^{N+1}}{2N-1}$

FWIW: after $N=10^5$ terms

$\pi \approx 3.1415\underline{\underline{8}}$ pretty

terrible for so much work

$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 \ldots$

$\int \frac{1}{1+x^2}dx = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$

$\text{Tan}(\pi/4) = 1$ so $\text{Arctan}(1) = \frac{\pi}{4}$

$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \ldots$

$\pi \approx 4\left[1 - \frac{1}{3} + \frac{1}{5} \ldots\right]$

6) Let $A = \begin{pmatrix} 1 & i \\ 1 & 1+i \end{pmatrix}$

Show that $A^{-1} = \begin{pmatrix} 1+i & -i \\ -1 & 1 \end{pmatrix}$

Verify that $AA^{-1} = A^{-1}A = I$

$$\left[\begin{array}{cc|cc} 1 & i & 1 & 0 \\ 1 & 1+i & 0 & 1 \\ \hline 1 & i & 1 & 0 \\ 0 & 1 & -1 & 1 \\ \hline 1 & 0 & 1+i & -i \\ 0 & 1 & -1 & 1 \end{array}\right] \checkmark$$

$$\begin{bmatrix} 1 & i \\ 1 & 1+i \end{bmatrix}\begin{bmatrix} 1+i & -i \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1+i & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & i \\ 1 & 1+i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

7) The matrix

$$\sigma_2 = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}$$

is a **Dirac** matrix. It appears in quantum electro-dynamics

1) show that σ_2 is unitary
2) what are the E-values of σ_2.
3) show that $\begin{pmatrix} 0 \\ 0 \\ i \\ 1 \end{pmatrix}$ is an E-vector.

1) easy
2) The charac. polynomial is

$$\lambda^4 - 2\lambda^2 + 1 = 0$$

quadratic in $\rho = \lambda^2$

$$\rho^2 - 2\rho + 1 = 0$$

$$\rho = \frac{2 \pm \sqrt{4-4}}{2} = 1 \quad \underline{\text{repeated}}$$

$\lambda^2 = 1$ $\quad \lambda = \pm 1$

$(\pm 1)^2 = 1$ $\quad (\pm 1)^2 = 1$

So the 4 E-values are

1, 1, -1, -1.

$$\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -i \\ -1 \end{pmatrix}$$

E-value of -1

8) What is the transformation matrix that maps the 2 vectors $\begin{pmatrix}1\\0\end{pmatrix}$ $\begin{pmatrix}0\\1\end{pmatrix}$ into, respectively, $\begin{pmatrix}-1\\0\end{pmatrix}$ $\begin{pmatrix}0\\-1\end{pmatrix}$.

is this matrix a rotation matrix? proove it.

Now do the same for the mapping that maps the vectors

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \longrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\-1\end{pmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \longrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

$-a = 1 \quad -b = 0$

$-c = 0 \quad -d = -1$

$$\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}-1 & 1\\-1 & -1\end{pmatrix}$$

$$\left.\begin{aligned} a+b &= -1 \\ a-b &= 1 \end{aligned}\right\} \qquad a=0$$

$$\left.\begin{aligned} c+d &= -1 \\ c-d &= -1 \end{aligned}\right\} \qquad 2c=-2$$

$$2b=-2$$

$$2d=0$$

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

9) Let $A = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$

compute the ε-values of A. Based on this is A diagonalizable?

$$(2-\lambda)(-\lambda) + 1 = 0$$

$$\lambda^2 - 2\lambda + 1 = 0$$

$$\lambda_\pm = \frac{2 \pm \sqrt{4-4}}{2} = 1$$

repeated ε-values. not diagonalizable.

10) compute the ε-values of ε-vectors of

$$\begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}$$

$$(3-\lambda)(-\lambda) + 1 = 0$$

$$\lambda^2 - 3\lambda + 1 = 0$$

$$\lambda_\pm = \frac{3 \pm \sqrt{5}}{2}$$

$$\vec{r}_1 = \frac{1}{2}\begin{pmatrix} 3+\sqrt{5} \\ 2 \end{pmatrix} \qquad \vec{r}_2 = \frac{1}{2}\begin{pmatrix} 3-\sqrt{5} \\ 2 \end{pmatrix}$$

11) Let 1 grain of wheat be placed on the first square of a chessboard; 2 on the second; 4 on the third; 8 on the forth, etc. On an 8x8 chess-board, how many grains of wheat will there be?

square	1	2	3	4	....
grains	1	2	4	8	

$$S_n = \sum_{n=0}^{63} 2^n = \frac{1-2^{64}}{1-2}$$

$$= 2^{64} - 1$$

$$\approx 10^{19}$$

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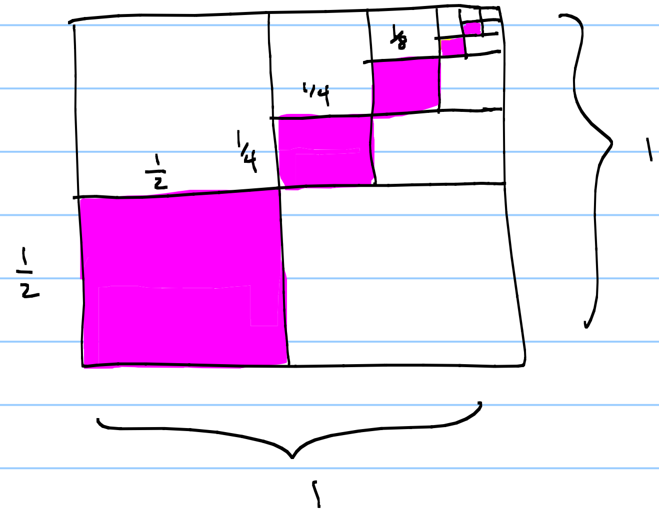

if this division is repeated indefinitely show that the total area colored in is $\frac{1}{3}$.

$$\text{Area} = \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{4}\right)^2 + \left(\tfrac{1}{8}\right)^2 + \cdots$$

$$= \left(\tfrac{1}{4}\right)^1 + \left(\tfrac{1}{4}\right)^2 + \left(\tfrac{1}{4}\right)^3 + \cdots$$

$$= \left[\underbrace{\left(\tfrac{1}{4}\right)^0 + \left(\tfrac{1}{4}\right)^1 + \left(\tfrac{1}{4}\right)^2 + \cdots}\right] - 1$$

$$= \frac{1}{1 - 1/4} - 1$$

$$= \frac{1}{3}$$