3D wave propagation

$$\nabla^2\mathbf{E} - \frac{n_j^{\ 2}}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E} = \frac{\partial^2}{\partial z^2}\mathbf{E} + \nabla_\perp^{\ 2}\mathbf{E} - \frac{n(\mathbf{r})^2}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E} = 0$$

$$\nabla_\perp^2 = \partial_x^2 + \partial_y^2 \qquad \boxed{\nabla_\perp^2 = \frac{1}{r}\partial_r\left(r\,\partial_r\right) + \frac{1}{r^2}\partial_\phi^2}$$

- Note:
  - All linear propagation effects are included in LHS: diffraction, interference, focusing…
  - With plane waves transverse derivatives are zero.
- More general examples:
  - Gaussian beams (including high-order)
  - Waveguides
  - Arbitrary propagation
  - Can determine discrete solutions to linear equation (e.g. Gaussian modes, waveguide modes), then express fields in terms of those solutions.

General 3D plane wave solution

- Assume separable function

$$\mathbf{E}(x,y,z,t) \sim f_1(x)\,f_2(y)\,f_3(z)\,g(t)$$

$$\vec{\nabla}^2\mathbf{E}(z,t) = \frac{\partial^2}{\partial x^2}\mathbf{E}(z,t) + \frac{\partial^2}{\partial y^2}\mathbf{E}(z,t) + \frac{\partial^2}{\partial z^2}\mathbf{E}(z,t) = \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E}(z,t)$$

- Solution takes the form:

$$\mathbf{E}(x,y,z,t) = \mathbf{E_0} e^{ik_x x} e^{ik_y y} e^{ik_z z} e^{-i\omega t} = \mathbf{E_0} e^{i\left(k_x x + k_y y + k_z z\right)} e^{-i\omega t}$$

$$\mathbf{E}(x,y,z,t) = \mathbf{E_0} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

  - Now k-vector can point in arbitrary direction

- With this solution in W.E.:

$$\boxed{n^2\frac{\omega^2}{c^2} = k_x^2 + k_y^2 + k_z^2 = \mathbf{k}\cdot\mathbf{k}}$$

Valid even in waveguides and resonators

Grad and curl of 3D plane waves

- Simple trick:

$$\nabla \cdot \mathbf{E} = \partial_x E_x + \partial_y E_y + \partial_z E_z$$

  – For a plane wave,

$$\nabla \cdot \mathbf{E} = i\left(k_x E_x + k_y E_y + k_z E_z\right) = i\left(\mathbf{k} \cdot \mathbf{E}\right)$$

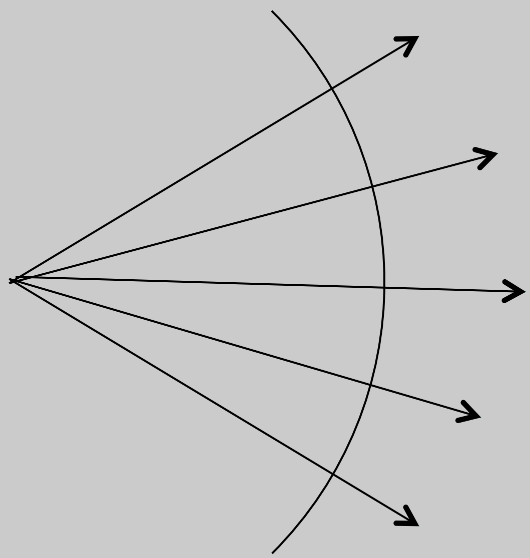
  – Similarly,

$$\nabla \times \mathbf{E} = i\left(\mathbf{k} \times \mathbf{E}\right)$$

- Consequence: since $\nabla \cdot \mathbf{E} = 0, \quad \mathbf{k} \perp \mathbf{E}$
  – For a given k direction, E lies in a plane
  – E.g. x and y linear polarization for a wave propagating in z direction

Curved wavefronts

- Rays are directed normal to surfaces of constant phase
  - These surfaces are the wavefronts
  - Radius of curvature is approximately at the focal point



- Spherical waves are approximate solutions to the wave equation (away from r = 0)

$$\nabla^2 E + \frac{n^2\omega^2}{c^2} E = 0$$

$$E \propto \frac{1}{r} e^{i(\pm kr - \omega t)}$$

Scalar *r*
+ outward
- inward

$$I \propto \frac{1}{r^2}$$

Paraxial approximations

- For **rays**, paraxial = small angle to optical axis
  - Ray slope: $\tan\theta \approx \theta$
- For **spherical waves** where power is directed forward:

$$e^{ikr} = \exp\left[ik\sqrt{x^2+y^2+z^2}\right]$$

$$k\sqrt{x^2+y^2+z^2} = kz\sqrt{1+\frac{x^2+y^2}{z^2}} \approx kz\left(1+\frac{x^2+y^2}{2z^2}\right)$$

Expanding to 1st order

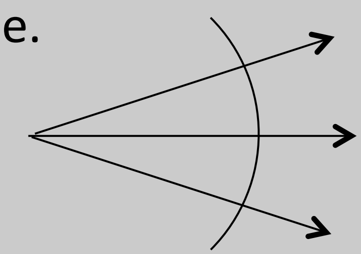
$$e^{i(kr-\omega t)} \to e^{ikz}\exp\left[i\left(k\frac{x^2+y^2}{2z}-\omega t\right)\right]$$

z is radius of curvature

Wavefront = surface of constant phase

$$k\frac{x^2+y^2}{2z} = \omega t$$

For x, y >0, t must increase.
Wave is diverging:

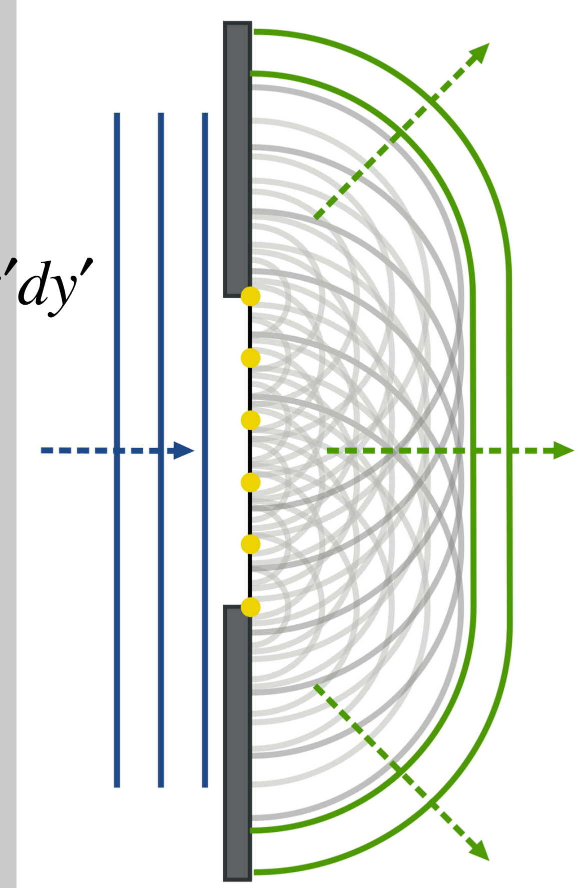


Diffractive propagation

- Huygens' principle:
  - Represent a plane wave as a superposition of source points emitting spherical waves
- Integral representation:

$$E(x,y,z)=\frac{i}{\lambda}\iint \underbrace{E(x',y',z')}_{\text{Field at input plane}}\underbrace{\frac{\exp\left[-ik\left|\mathbf{r}-\mathbf{r}'\right|\right]}{\left|\mathbf{r}-\mathbf{r}'\right|}}_{\text{Spherical wavelet}}\underbrace{\cos\theta}_{\text{Inclination factor}}dx'dy'$$

This is essentially a convolution of the complex input field with the spherical wavelets, which are the Green's function for the wave equation



Maxwell's Equations to wave eqn

- Write Maxwell's eqns for a linear medium

$$\vec{\nabla}\cdot\left(\varepsilon_0\varepsilon\mathbf{E}\right)=0 \qquad \vec{\nabla}\times\mathbf{E}=-\mu_0\mu\frac{\partial\mathbf{H}}{\partial t}$$

$$\vec{\nabla}\cdot\left(\mu_0\mu\mathbf{H}\right)=0 \qquad \vec{\nabla}\times\mathbf{H}=\varepsilon_0\varepsilon\frac{\partial\mathbf{E}}{\partial t}$$

- Assume:
  - Non-magnetic medium ($\mu = 0$)
  - Linear medium $D = \varepsilon_0\ \varepsilon\ \mathbf{E}$
  - Non-dispersive medium

Take the curl:

$$\vec{\nabla}\times\left(\vec{\nabla}\times\mathbf{E}\right)=-\mu_0\frac{\partial}{\partial t}\vec{\nabla}\times\mathbf{H}=-\mu_0\frac{\partial}{\partial t}\left(\varepsilon_0\varepsilon\frac{\partial\mathbf{E}}{\partial t}\right)=-\frac{1}{c^2}\frac{\partial}{\partial t}\left(\varepsilon\frac{\partial\mathbf{E}}{\partial t}\right)$$

$$\vec{\nabla}\times\left(\vec{\nabla}\times\mathbf{E}\right)=\vec{\nabla}\left(\vec{\nabla}\cdot\mathbf{E}\right)-\left(\vec{\nabla}\cdot\vec{\nabla}\right)\mathbf{E}$$ BAC-CAB vector ID

Wave equation for spatially varying media

- Generalized wave equation

$$\vec{\nabla}\left(\vec{\nabla}\cdot\mathbf{E}\right)-\left(\vec{\nabla}\cdot\vec{\nabla}\right)\mathbf{E}=-\frac{1}{c^2}\frac{\partial}{\partial t}\left(\varepsilon\frac{\partial\mathbf{E}}{\partial t}\right)=-\frac{1}{c^2}\varepsilon\frac{\partial^2\mathbf{E}}{\partial t^2}$$

If ε is time-independent

  - If medium has a spatially-varying refractive index:

$$\vec{\nabla}\cdot\left(\varepsilon\mathbf{E}\right)=\varepsilon\vec{\nabla}\cdot\mathbf{E}+\left(\mathbf{E}\cdot\vec{\nabla}\right)\varepsilon=0 \qquad \rightarrow\vec{\nabla}\cdot\mathbf{E}=-\frac{1}{\varepsilon}\left(\mathbf{E}\cdot\vec{\nabla}\right)\varepsilon=-\left(\mathbf{E}\cdot\vec{\nabla}\right)\ln\varepsilon$$

$$\vec{\nabla}^2\mathbf{E}+\vec{\nabla}\left(\left(\mathbf{E}\cdot\vec{\nabla}\right)\ln\varepsilon\right)-\frac{1}{c^2}\varepsilon\frac{\partial^2\mathbf{E}}{\partial t^2}=0$$

  - Use above for P polarized light (E has component along gradient.
  - For S polarization or no gradient, eliminate blue term.

Helmholtz (scalar) equation

- We will ignore vector components of field
  - S polarization or no strong index gradients
  - No boundary conditions (e.g. waveguides)
  - Some limit on angular range, tight focusing

$$\nabla^2 U - \frac{\varepsilon}{c^2}\frac{\partial^2}{\partial t^2}U = 0$$

  - Monochromatic (for now)

$$\nabla^2 U + \frac{\varepsilon}{c^2}\omega^2 U = \nabla^2 U + k^2 U = 0$$

This is an equation for source-free wave propagation

  - Green's function satisfies

$$\left(\nabla^2 + k^2\right)G = -\delta\left(\mathbf{r} - \mathbf{r}'\right)$$

This adds a δ-fcn source
G(r) is the wave emitted from this point source.

Green's theorem

- Mathematical basis for diffraction
- Start with divergence theorem

$$\int_V \nabla \cdot \mathbf{A}\, dV = \oint_S \mathbf{A} \cdot \mathbf{n}\, da$$

- Let $\mathbf{A} = \psi \nabla \chi$ with ψ, χ scalar functions

$$\int_V \nabla \cdot \mathbf{A}\, dV = \int_V \nabla \cdot (\psi \nabla \chi)\, dV = \int_V \left[\psi \nabla^2 \chi + (\nabla \psi) \cdot (\nabla \chi)\right] dV$$

$$\oint_S \mathbf{A} \cdot \hat{\mathbf{n}}\, da = \oint_S (\psi \nabla \chi) \cdot \hat{\mathbf{n}}\, da = \oint_S \psi \frac{\partial \chi}{\partial n} da$$

Gradient only in direction normal to surface
n *points out from surface*

- Now interchange ψ, χ then subtract

$$\oint_S \left(\psi \frac{\partial \chi}{\partial n} - \chi \frac{\partial \psi}{\partial n} \right) da = \int_V \left[\psi \nabla^2 \chi - \chi \nabla^2 \psi \right] dV$$

Let $\psi \rightarrow U, \chi \rightarrow G$
G = Green's function

Using Green's function for wave equation

- For linear differential equation, put δ(x-x') as source term. G(x) is effectively impulse response.
- Get answer for general inhomogeneous function by convolving G with source distribution
- Different choices of G are possible (assess accuracy)
  - Kirchhoff:
  - Ideal spherical wave
  - Discontinuity at origin
  - Let S' = S + S_ε , then take limit small ε
  - This excludes source point, so inside V'

$$G(P_1) = \frac{e^{ikr_{01}}}{r_{01}}$$

$$\left(\nabla^2 + k^2\right)G = 0 \rightarrow \nabla^2 G = -k^2 G$$

$$\left(\nabla^2 + k^2\right)U = 0 \rightarrow \nabla^2 U = -k^2 U$$

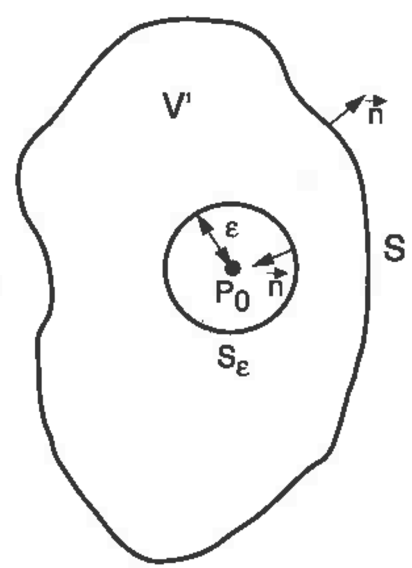


Figure 3.5 Surface of integration.

Computing diffraction integral

- Green's thm:

$$\oint_{S'}\left(U\frac{\partial G}{\partial n}-G\frac{\partial U}{\partial n}\right)da=\int_V\left[U\nabla^2 G-G\nabla^2 U\right]dV=\int_V\left[-Uk^2G+Gk^2U\right]dV=0$$

$$\nabla^2 G=-k^2 G \quad \nabla^2 U=-k^2 U$$

  - For separate regions, $S'=S+S_\varepsilon$

$$-\oint_{S_\varepsilon}\left(U\frac{\partial G}{\partial n}-G\frac{\partial U}{\partial n}\right)ds=\oint_{S}\left(U\frac{\partial G}{\partial n}-G\frac{\partial U}{\partial n}\right)ds$$

  - For a point in S',

$$G(P_1)=\frac{e^{ikr_{01}}}{r_{01}} \quad \rightarrow \frac{\partial G(P_1)}{\partial n}=\cos\left(\hat{\mathbf{n}},\mathbf{r_{01}}\right)\left(ik-\frac{1}{r_{01}}\right)\frac{e^{ikr_{01}}}{r_{01}}$$

  - If the point P_1 is on S_ε, $\cos\left(\hat{\mathbf{n}},\mathbf{r_{01}}\right)=-1$

$$G(P_1)=\frac{e^{ik\varepsilon}}{\varepsilon} \quad \rightarrow \frac{\partial G(P_1)}{\partial n}=\left(\frac{1}{\varepsilon}-ik\right)\frac{e^{ik\varepsilon}}{\varepsilon}$$

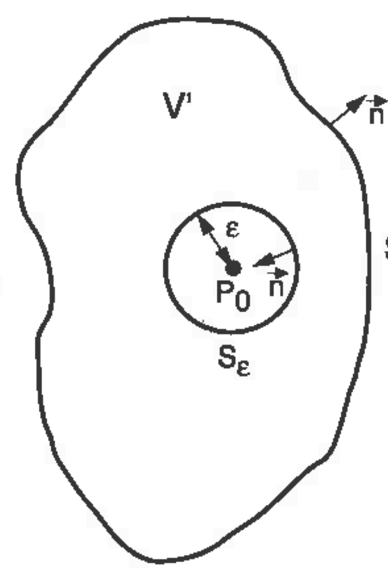


Figure 3.5 Surface of integration.

Helmholtz/Kirchhoff diffraction integral

- Take the limit of arbitrarily small ε

$$\lim_{\varepsilon\to 0}\oint_{S_\varepsilon}\left(U\frac{\partial G}{\partial n}-G\frac{\partial U}{\partial n}\right)ds=\lim_{\varepsilon\to 0}4\pi\varepsilon^2\left[U\left(P_0\right)\left(\frac{1}{\varepsilon}-ik\right)\frac{e^{ik\varepsilon}}{\varepsilon}-\frac{e^{ik\varepsilon}}{\varepsilon}\frac{\partial U\left(P_0\right)}{\partial n}\right]$$

$$=4\pi\lim_{\varepsilon\to 0}\left[U\left(P_0\right)\left(1-ik\varepsilon\right)e^{ik\varepsilon}-\varepsilon e^{ik\varepsilon}\frac{\partial U\left(P_0\right)}{\partial n}\right]=4\pi U\left(P_0\right)$$

- Now put this into the Green's function surface integral

$$\oint_{S_\varepsilon}\left(U\frac{\partial G}{\partial n}-G\frac{\partial U}{\partial n}\right)ds=\oint_{S}\left(G\frac{\partial U}{\partial n}-U\frac{\partial G}{\partial n}\right)ds$$

$$U\left(P_0\right)=\frac{1}{4\pi}\oint_{S}\left(\frac{e^{ikr_{01}}}{r_{01}}\frac{\partial U}{\partial n}-U\frac{\partial}{\partial n}\left(\frac{e^{ikr_{01}}}{r_{01}}\right)\right)ds$$

- The field at P_0 can be determined by integrating around any surface that surrounds it.

Diffraction by a plane screen

$$U(P_0) = \frac{1}{4\pi}\oint_{S_1+S_2}\left(G\frac{\partial U}{\partial n} - U\frac{\partial G}{\partial n}\right)ds$$

Intuitively, we don't expect much contribution from S_2 : assume only *outgoing* waves on this surface.

Kirchhoff approach:
Assume field is incident from left on S_1

- U and dU/dn are the same as incident
- no contribution from opaque region outside opening Σ.
- Integrate only over Σ

Trouble: this restriction ends up being unphysical. Leads to alternative choices of Green's functions

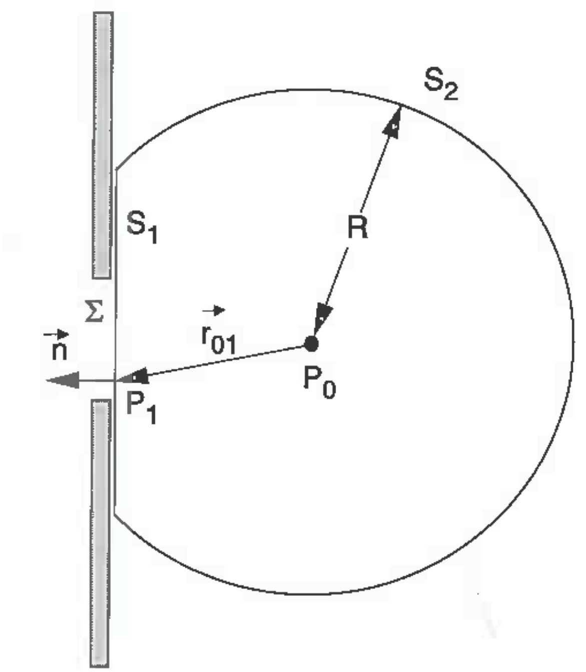


Figure 3.6 Kirchhoff formulation of diffraction by a plane screen.

Very little distinction between approaches when $r_{01} \gg \lambda$

Diffraction formulas

- Kirchhoff

$r_{01} >> \lambda$

$$G(P_1) = \frac{e^{ikr_{01}}}{r_{01}} \quad \rightarrow \frac{\partial G(P_1)}{\partial n} = \cos(\hat{\mathbf{n}}, \mathbf{r_{01}}) \left(ik - \frac{1}{r_{01}} \right) \frac{e^{ikr_{01}}}{r_{01}} \approx ik \cos(\hat{\mathbf{n}}, \mathbf{r_{01}}) \frac{e^{ikr_{01}}}{r_{01}}$$

$$U(P_0) = \frac{1}{4\pi} \oint_\Sigma \left(G \frac{\partial U}{\partial n} - U \frac{\partial G}{\partial n} \right) ds = \frac{1}{4\pi} \oint_\Sigma \left(\frac{\partial U}{\partial n} - ikU \cos(\hat{\mathbf{n}}, \mathbf{r_{01}}) \right) \frac{e^{ikr_{01}}}{r_{01}} ds$$

- For illumination of Σ by a point source at P_2,

$$U(P_1) = A \frac{e^{ikr_{21}}}{r_{21}}$$

$$\boxed{U(P_0) = \frac{A}{i\lambda} \oint_\Sigma \left(\frac{\cos(\hat{\mathbf{n}}, \mathbf{r_{01}}) - \cos(\hat{\mathbf{n}}, \mathbf{r_{21}})}{2} \right) \frac{e^{ik(r_{01}+r_{21})}}{r_{01} r_{21}} ds}$$

- Sommerfeld: avoid unphysical constraint on U

$$\boxed{U(P_0) = \frac{A}{i\lambda} \oint_\Sigma \cos(\hat{\mathbf{n}}, \mathbf{r_{01}}) \frac{e^{ik(r_{01}+r_{21})}}{r_{01} r_{21}} ds} \qquad \boxed{U(P_0) = -\frac{A}{i\lambda} \oint_\Sigma \cos(\hat{\mathbf{n}}, \mathbf{r_{21}}) \frac{e^{ik(r_{01}+r_{21})}}{r_{01} r_{21}} ds}$$

Obliquity factors modifying Huygens

- The diffraction equations are generated by mathematical constructs that help solve the wave equation
  - Adapt 1st Sommerfeld equation:

$$\boxed{U(P_0)=\frac{A}{i\lambda}\oint_{\Sigma}\frac{e^{ikr_{21}}}{r_{21}}\frac{e^{ikr_{01}}}{r_{01}}\cos(\hat{\mathbf{n}},\mathbf{r_{01}})\,ds \rightarrow \frac{A}{i\lambda}\oint_{\Sigma}U(P_1)\frac{e^{ikr_{01}}}{r_{01}}\psi(\theta)\,ds}$$

  - Extra added function: obliquity factor $\rightarrow$ plane wave incident
  - Kirchhoff $\boxed{\psi=\tfrac{1}{2}\left(\cos(\hat{\mathbf{n}},\mathbf{r}_{01})-\cos(\hat{\mathbf{n}},\mathbf{r}_{21})\right)\rightarrow\tfrac{1}{2}(1+\cos\theta)}$
  - Sommerfeld 1 $\boxed{\psi=\cos(\hat{\mathbf{n}},\mathbf{r}_{01})\rightarrow\cos\theta}$
  - Sommerfeld 2 $\boxed{\psi=-\cos(\hat{\mathbf{n}},\mathbf{r}_{21})\rightarrow 1}$
- If we had real dipole emitters:

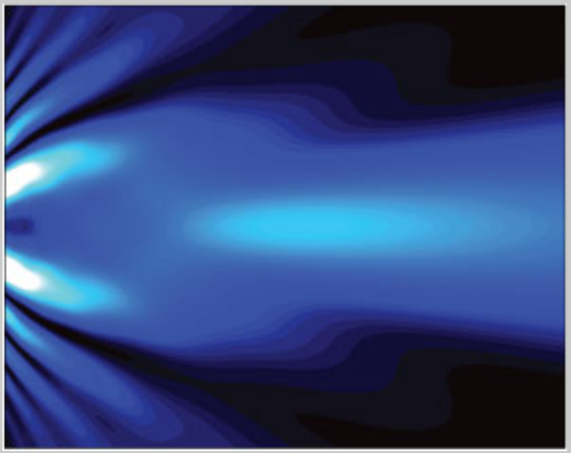
$$E\sim\frac{e^{ikr}}{r}\cos\theta$$

But here, θ is measured from oscillator direction.

Constraints on diffraction integrals

- These integrals are very accurate in many practical situations but…
  - R >> λ: the approach doesn't work for "near-field" situations, e.g. NSOM, nanophotonics, RF or THz waves where the structures are close in size to the wavelength.
  - Boundary conditions, contributions from screen surface: sometimes the physical nature of the screen can be important. Example: surface plasmon waves can be excited on metal surfaces, propagate through hole, then be re-radiated on other side.
  - Metamaterials, photonic crystals…

Use RF approaches to directly solve Maxwell equations in these cases.



Paraxial, slowly-varying approximations

- Assume
  - waves are forward-propagating:
    $\mathbf{E}(\mathbf{r},t)=\mathbf{A}(\mathbf{r})e^{i(kz-\omega_0 t)}+\text{c.c.}$
  - Refractive index is isotropic

$$\frac{\partial^2}{\partial z^2}\mathbf{A}+2ik\frac{\partial}{\partial z}\mathbf{A}-k^2\mathbf{A}+\nabla_\perp{}^2\mathbf{A}+\frac{n^2\omega_0{}^2}{c^2}\mathbf{A}=0$$

  - Fast oscillating carrier terms cancel (blue)
- Slowly-varying envelope: compare red terms
  - Drop 2nd order deriv if $\frac{2\pi}{\lambda}\frac{1}{L}A\gg\frac{1}{L^2}A$
  - This ignores:
    - Changes in z as fast as the wavlength
    - Counterpropagating waves

$$\boxed{2ik\frac{\partial}{\partial z}\mathbf{A}+\nabla_\perp{}^2\mathbf{A}=0}$$

Fresnel diffraction integral

- Fresnel approximation (near field)
  - Expand the spherical wave in paraxial approximation (in exponential)
  - Let denominator be $|\mathbf{r}-\mathbf{r}'| \sim z - z' = L \quad \cos\theta \simeq 1$
  - Input field: $E(x',y',z') = u(x',y',z')e^{-ik(z-z')}$

$$u(x,y,z) = \frac{i}{\lambda L}\iint u(x',y',z')\exp\left[-ik\frac{(x-x')^2+(y-y')^2}{2L}\right]dx'dy'$$

$$u(x,y,z) = \frac{i}{\lambda L}\mathrm{e}^{-ik\frac{x^2+y^2}{2L}}\iint u(x',y',z')\mathrm{e}^{-ik\frac{x'^2+y'^2}{2L}}\,\mathrm{e}^{-i\frac{k}{L}(xx'+yy')}\,dx'dy'$$

Fraunhofer diffraction integral

$$u(x,y,z) = \frac{i}{\lambda L} \mathrm{e}^{-ik\frac{x^2+y^2}{2L}} \iint u(x',y',z') \mathrm{e}^{-ik\frac{x'^2+y'^2}{2L}} \mathrm{e}^{-i\frac{k}{L}(xx'+yy')} dx'dy'$$

- In the "far field", we approximate the sum of paraxial spherical waves as a sum of plane waves
  - Assume field in input plane is confined to a radius *a*
  - If $\frac{ka^2}{2L} = \frac{\pi a^2}{\lambda} \frac{1}{L} << 1$ then we drop quadratic phases.

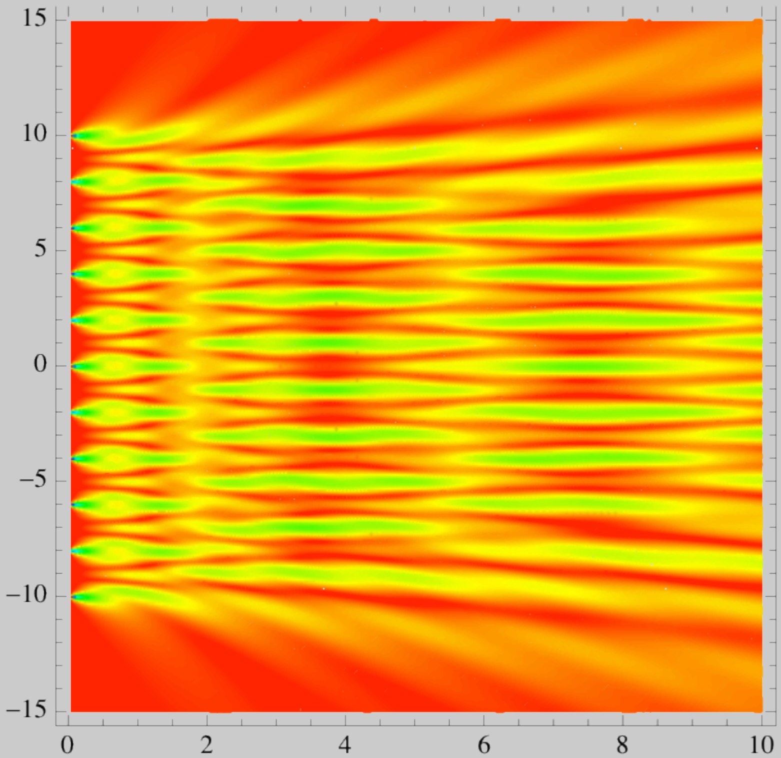
$$u(x,y,z) = \frac{i}{\lambda L} \iint u(x',y',z') \exp\left[-i\left(\frac{kx}{L} x' + \frac{ky}{L} y'\right)\right] dx'dy'$$

  - Result: far field is a Fourier transform of the input field
  - "spatial frequencies" $\beta_x = k\frac{x}{L} = k\sin\theta_x \quad \beta_y = k\frac{y}{L} = k\sin\theta_y$

Example: sum of dipole radiators

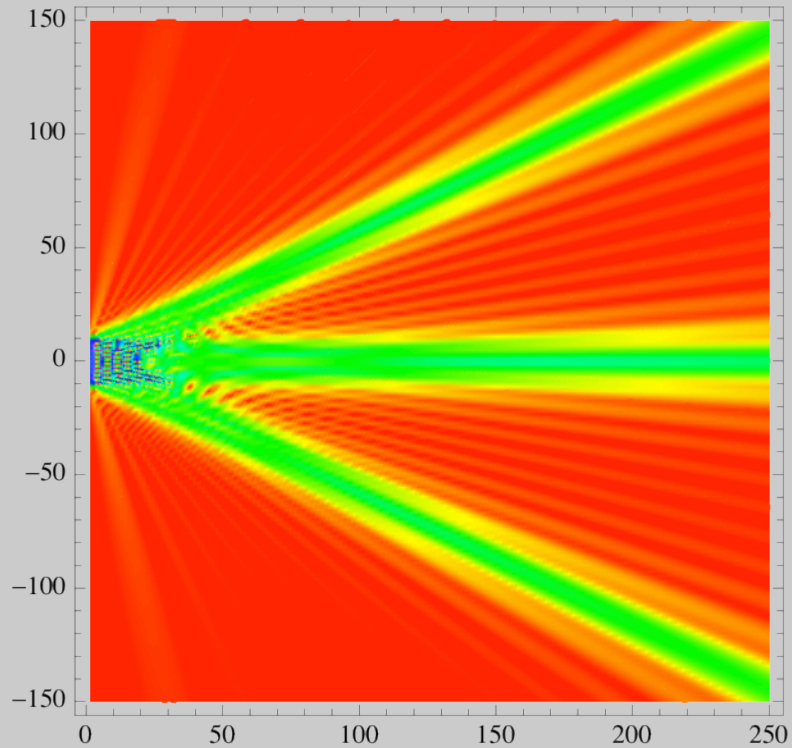
- Add fields from 10 individual sources

Near field



Talbot fringes

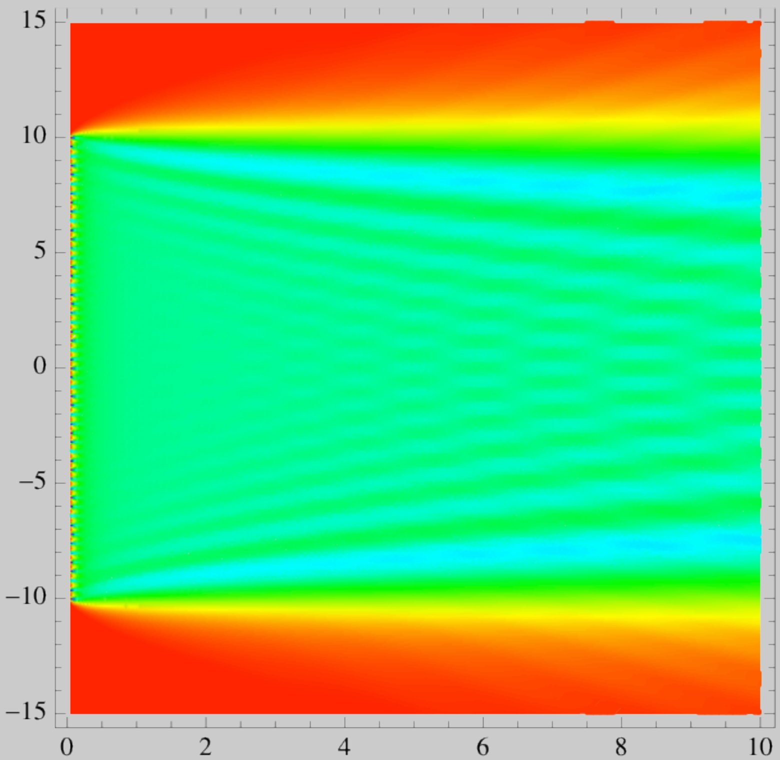
far field



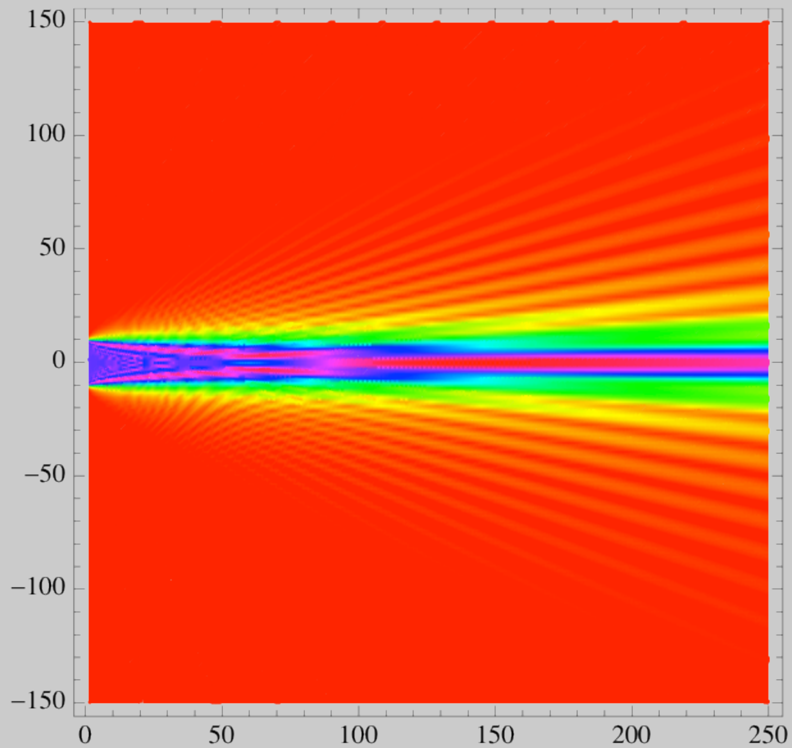
Diffraction grating

High-density of radiators

- Combine 50 sources over same distance



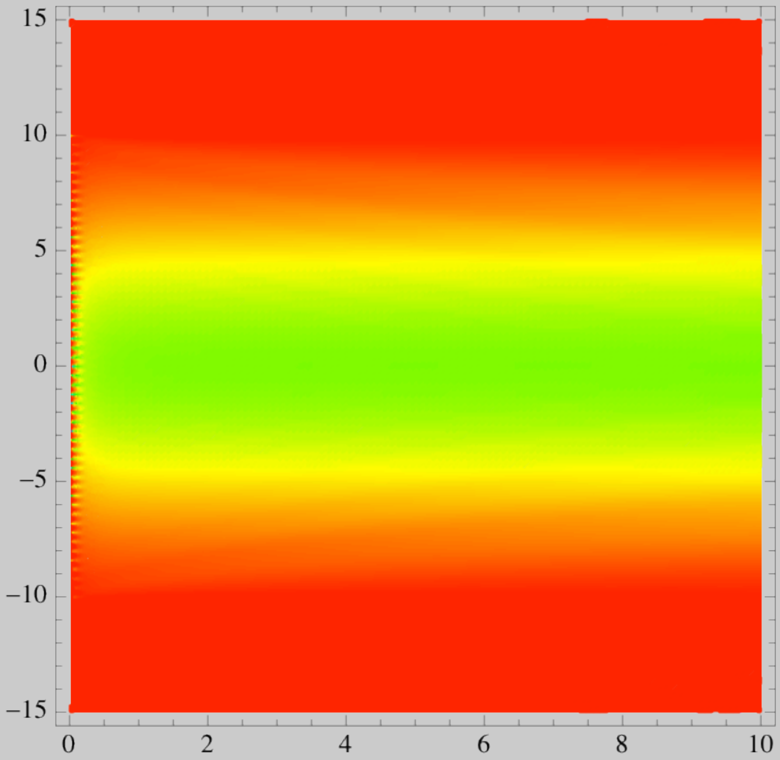
Fresnel zone shows shadow boundary, diffraction fringes

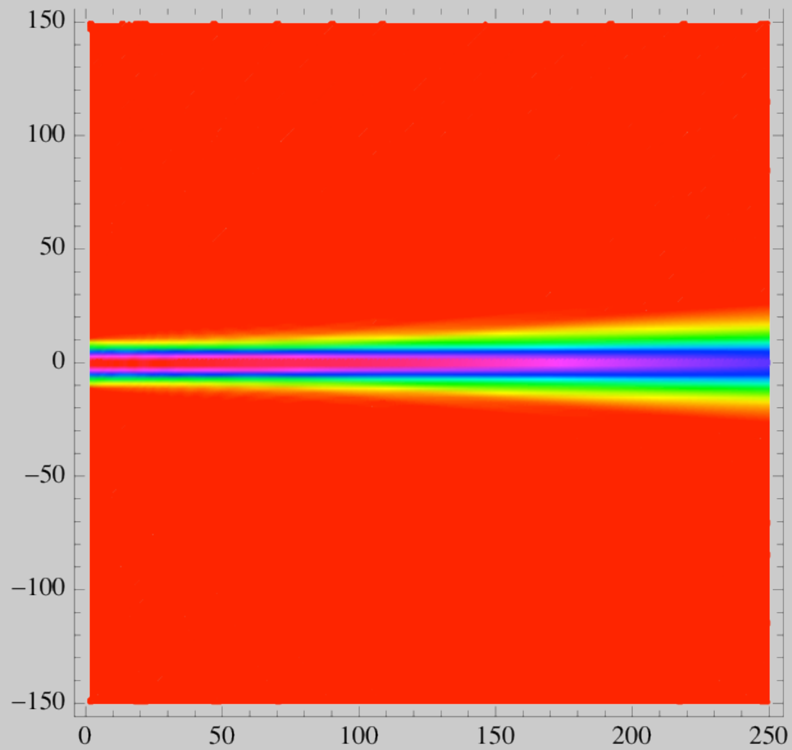


Far field evolves more like a beam, with single-slit diffraction.

High density of radiators, Gaussian envelope

- Gaussian amplitude envelope eliminates diffraction fringes





Beam smoothly spreads out with distance