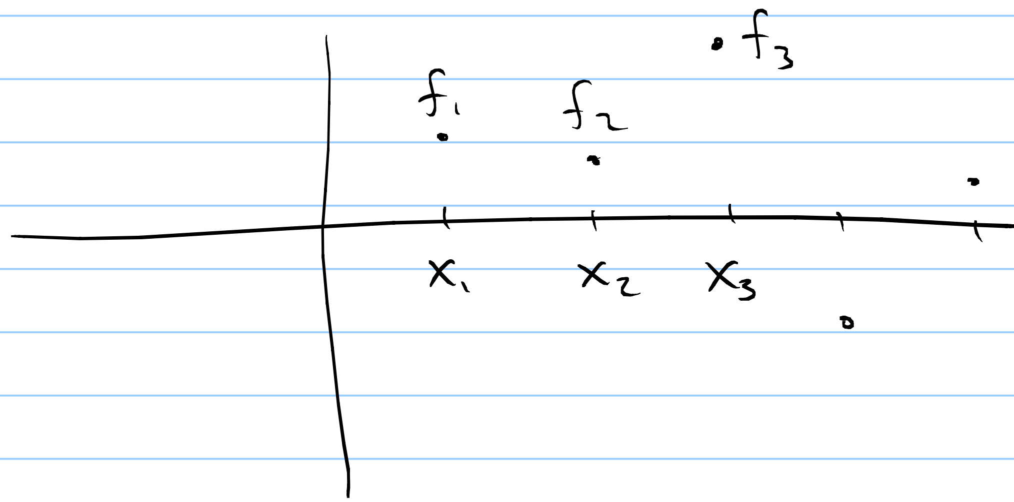
11-7-07 ①

Note Title 11/6/2006

Trig. interpolation of data:

$$(f_k, x_k)$$



$$P(x) = \sum_{n=0}^{N-1} c_n e^{inx}$$

polynomial in e^{ix}

interpolation means

$$\boxed{P(x_k) = f_k}$$

②

Hence

$$f_k = \sum_{n=0}^{N-1} c_n e^{i n \overset{p(t_k)}{x_k}}$$

we said $x_k = \dfrac{2\pi k}{N}$

$$\text{so } f_k = \sum_{n=0}^{N-1} c_n \underbrace{e^{i 2\pi n k / N}}_{\text{matrix}_{(k,n)}}$$

call this matrix Q

$$\left(Q^{*T} Q\right)_{kl} = \sum_{j=0}^{N-1} \left(Q^{*T}\right)_{kj} Q_{jl}$$

$Q^{*T} = Q^{+} =$ Hermition of Q

$$Q^{\dagger} Q = \sum_{j=0}^{N-1} e^{-i 2\pi k j/N} \, e^{i 2\pi j l/N}$$

$$= \sum_{j=0}^{N-1} e^{i 2\pi j (l-k)/N}$$

$$= \begin{cases} 0 & \text{if } l \neq k \\ \sum_{j=0}^{N-1} 1 & \text{if } l = k \end{cases}$$

$$\sum_{j=0}^{N-1} 1 = N$$

This relies on the fact that

$$\boxed{\sum_{j=0}^{N-1} \cos(2\pi \alpha j) = 1 \quad \text{if } 0 < \alpha < 1}$$

4

So if we normalize Q by $\frac{1}{\sqrt{N}}$ then

$$Q^{*T} Q = \mathbb{I}$$

So $\vec{f} = Q \vec{c}$

$$\Rightarrow \vec{c} = Q^{*T} \vec{f}$$

$$= Q^{+} \vec{f} \quad \text{DFT}$$

interpolation coefficients

data

5

$$\boxed{\vec{c} = Q^{+}\vec{f}}$$

Discrete Fourier Transform examples

$$\vec{c} = Q^{+}\vec{f}$$

$$c_k = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f_n e^{-2\pi i n k/N}$$

You could easily program this.

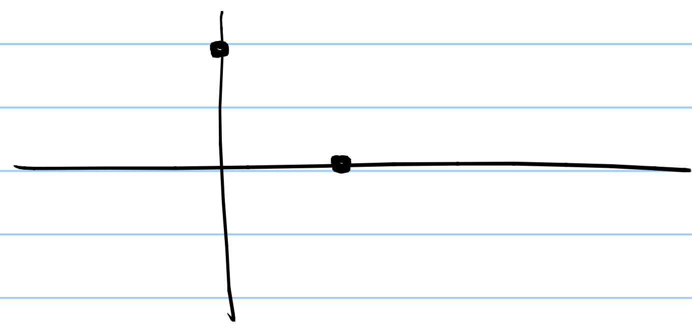
Algorithm

1) loop over all k from $0 \rightarrow N-1$

6

2) For each k do the sum

E.g.



$f_0 = 1 \qquad f_1 = 0$

$$C_k = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f_n e^{-2\pi i n k/N}$$

$$C_k = \frac{1}{\sqrt{2}} \left[f_0 e^{0} + f_1 e^{-\pi i k} \right]$$

$$C_0 = \frac{1}{\sqrt{2}} \left[1 \cdot e^{-0} + 0 \right]$$

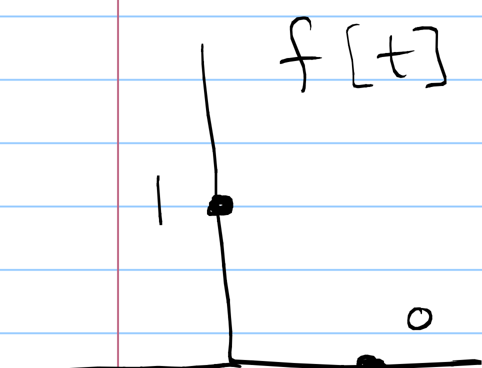
$$C_1 = \frac{1}{\sqrt{2}} \left[1 \cdot e^{-0} + 0 \right]$$

⑦ So the DFT of the "time series" $\{1, 0\}$ is

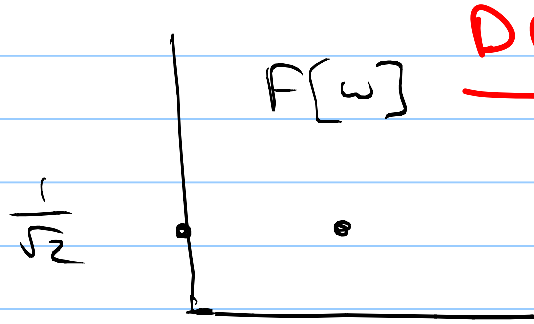
$$\left\{ \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\}$$

This is purely real

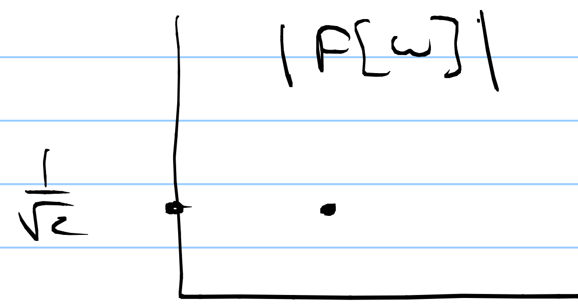
So the periodogram is equal to the DFT



DFT



periodogram



8

Ex. $f = \{0, 1, 0\}$

$$C_k = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f_n e^{-2\pi i n k/N}$$

only $n=1$ term of each sum is non zero

$$C_k = \frac{1}{\sqrt{3}} f_1 e^{-2\pi i k/3}$$

$$c_0 = \frac{1}{\sqrt{3}} \left[1 \, e^{-2\pi i 1 \cdot 0/N} \right] = .577$$

$$c_1 = \frac{1}{\sqrt{3}} \left[1 \, e^{-2\pi i/3} \right]$$

$$\mathrm{Re}[c_1] = \frac{\cos(2\pi/3)}{\sqrt{3}} = -.288$$

9

$$\text{Im}[c_1] = \frac{\sin(2\pi/3)}{\sqrt{3}} = .5$$

$$c_2 = \frac{1}{\sqrt{3}}\left[1 \cdot e^{-2\pi i\, 2/3}\right]$$

$$\text{Re}[c_2] = \frac{\cos(4\pi/3)}{\sqrt{3}} = -.288$$

$$\text{Im}[c_2] = \frac{\sin(4\pi/3)}{\sqrt{3}} = -.5$$

$$\vec{f} = \{0, 1, 0\}$$

$$\vec{c} = \{\underline{.577}, \underline{-.288} + \underline{.5}i, \underline{-.288} - \underline{.5}i\}$$

— re — imag

(10)

periodogram

$|\vec{c}| = \{.577, .577, .577\}$

The periodogram of a spike is flat no matter where in time the spike is located.

But the DFT of a spike is only flat if the spike is a t=0.

11

Try this with mathematica

X = {1, 1, 1, 1}

y = Fourier[X]

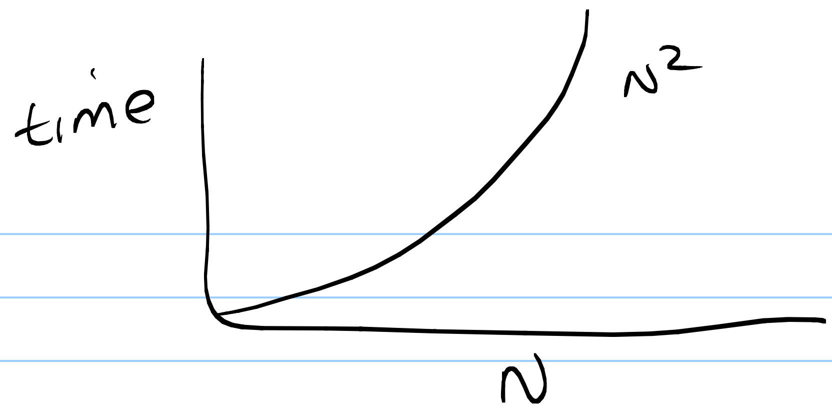
Chop[y] = {2, 0, 0, 0}

eliminates numerically zero parts

$$C_k = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f_n e^{-2\pi i n k/N}$$

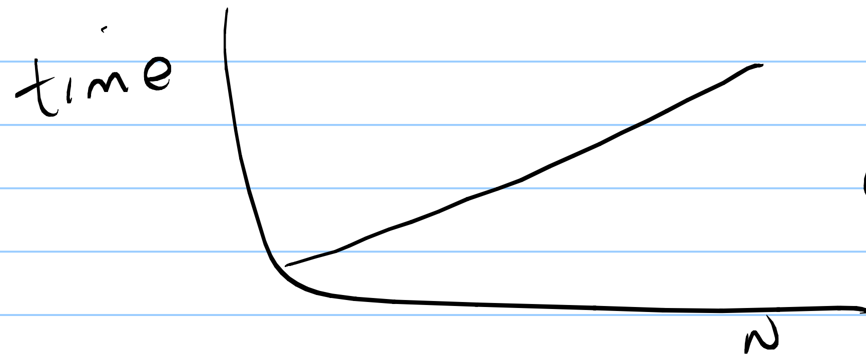
\# floating point operation

flop $O(N^2)$



DFT

$O(N^2)$



FFT

$O(N \log N)$

Gilbert Strang
"introduction to Linear Algebra"
Wellesley - Cambridge press
3rd edition
10.3 Fast Fourier Transform