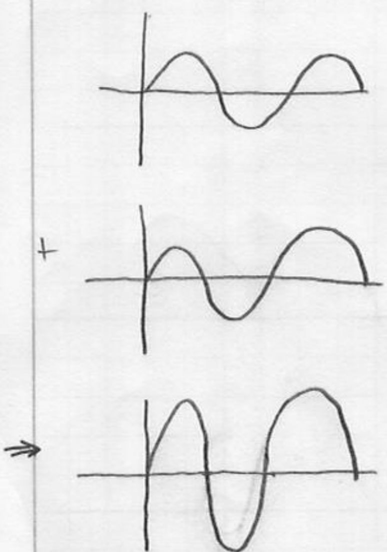
Intro to interference

From time to time we have more than one source of electromagnetic radiation, leading to the possibility of interference. The basic approach is simple enough: To find the net field somewhere, we add up all the individual fields. If we want net intensity, we can use those net fields to construct the Poynting vector.

$$\vec{E}_{net} = \vec{E}_1 + \vec{E}_2 + \ldots \quad , \quad I = |\vec{S}_{net}| = \frac{1}{\mu_0} |\vec{E}_{net} \times \vec{B}_{net}|$$

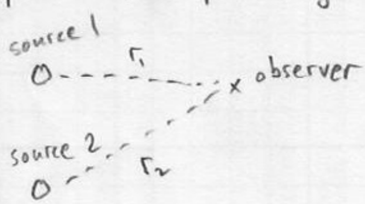
Back in intro physics, we kept it very simple. Given two fields that can be modeled as sines or cosines, if those sources were exactly in phase (and the same wavelength + polarization + amplitude) they'd add up to give twice the field and four times the intensity.



And two otherwise identical signals that were completely out of phase added up to nothing at all.

We called these constructive + destructive interference.

Back then, we came up with a general principle: Given two identical, originally in-phase sources, we'd get one kind of interference or another based on the relative distances between the sources and the observation point (the path length difference)



If the path length difference was an integer multiple of the radiation's wavelength, the light from the two sources would be in phase upon reaching the observer and we'd get constructive interference

$$r_2 - r_1 = m\lambda \quad , \quad \text{integer } m \Rightarrow \text{constructive}$$

Similarly, if the two sources ended up half a wavelength out of phase, we'd get destructive interference.

$$r_2 - r_1 = (m + 1/2)\lambda, \quad \text{integer } m \Rightarrow \text{destructive}$$

Now we're going to start relaxing some of those restrictions (same wavelength, amplitude, polarization, coherent sources), but not all at once.

Let's start with two sources (otherwise identical) arriving at some location out of phase with one another by some arbitrary angle δ. We'll use complex exponential notation.

$$E_1 = E_0 e^{i(kx-\omega t)} , \quad E_2 = E_0 e^{i(kx-\omega t+\delta)}$$

And we're observing these at the same location, so let's choose $x=0$

$$\rightarrow \quad E_1 = E_0 e^{-i\omega t} , \quad E_2 = E_0 e^{-i\omega t + i\delta} , \quad E_{net} = E_0\left(e^{-i\omega t} + e^{-i\omega t+i\delta}\right)$$

And for plane waves the corresponding B-fields would be

$$B_1 = E_0/c \, e^{-i\omega t} , \quad B_2 = E_0/c \, e^{-i\omega t+i\delta} , \quad B_{net} = E_0/c \left(e^{-i\omega t} + e^{-i\omega t+i\delta}\right)$$

To get the intensity, most properly we look at $|\vec{E}\times\vec{B}|/\mu_0$, but in this context we almost always want the time averaged intensity, and a shortcut to get that is $I = \frac{1}{2\mu_0} E B^*$

(the complex conjugate gets rid of the time dependence & complex goodies, and the $1/2$ puts in the $1/2$ we get from averaging terms like $\cos^2(\omega t)$ or $\sin^2(\omega t)$)

Thus, for the present example,

$$I = \frac{1}{2\mu_0}\frac{E_0^2}{c}\left(e^{-i\omega t} + e^{-i\omega t+i\delta}\right)\left(e^{+i\omega t} + e^{+i\omega t - i\delta}\right)$$

$$= \frac{1}{2\mu_0}\frac{E_0^2}{c}\left(1+1+e^{i\delta}+e^{-i\delta}\right) = \frac{1}{2\mu_0}\frac{E_0^2}{c}\, 2\left(1+\frac{e^{i\delta}+e^{-i\delta}}{2}\right)$$

$$= \frac{E_0^2}{\mu_0 c}\left(1+\cos(\delta)\right)$$

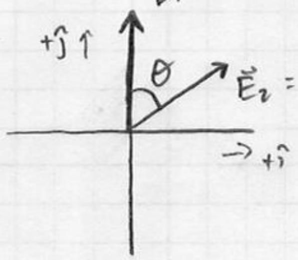
The base time averaged intensity of a single such source is

$I_0 = \frac{1}{2}\frac{E_0^2}{\mu_0 c}$, so we can write this as

$I = 2I_0\left(1+\cos(\delta)\right)$ Which maxes out at $4I_0$ when the sources are exactly in phase, and is a minimum of zero when the sources are out of phase.

Next, drop the assumption that the polarizations of your sources are the same. There's nothing too special about that. Take as an example:

$$\vec{E}_1 = E_0 e^{-i\omega t}\,\hat{j}$$



$$\vec{E}_2 = E_0 \cos\theta\, e^{-i\omega t + i\delta}\,\hat{j} + E_0 \sin\theta\, e^{-i\omega t + i\delta}\,\hat{i}$$

Two plane waves of the same amplitudes and frequencies, with one polarized at an angle theta with respect to the other, and also phase shifted by some amount δ.

Physically, we might reasonably expect that the component of $\vec{E}_2$ that's parallel to $\vec{E}_1$ will interfere, while the orthogonal component will come along for the ride. And that's what happens, but let's work it out to see what exactly that means.

$$\vec{E}_{net} = \vec{E}_1 + \vec{E}_2 = E_0 e^{-i\omega t}\left[\left(1+\cos\theta e^{i\delta}\right)\hat{j} + \sin\theta e^{i\delta}\,\hat{i}\right]$$

And it's from the intensity that we can really see what's going on. We can get that (time averaged) from:

$$I_{avg} = \frac{1}{2\mu_0}\left|\vec{E}\times\vec{B}^*\right|$$

And here we could go back and construct $\vec{B}_1$, $\vec{B}_2$, and $\vec{B}_{net}$ and do the cross product, but here's a trick:

These are plane waves, so $\vec{E}$ & $\vec{B}$ are perpendicular and their cross product has the max possible magnitude. Also, $B_0 = E_0/c$.

Thus $\left|\vec{E}\times\vec{B}^*\right| = \vec{E}\cdot\vec{E}^*/c$ That's much easier!

Continuing on,

$$I_{avg} = \frac{1}{2\mu_0 c}\,\vec{E}_{net}\cdot\vec{E}_{net}^*$$

$$= \frac{E_0^2}{2\mu_0 c}\left[\left(1+\cos\theta e^{i\delta}\right)\hat{j} + \sin\theta e^{i\delta}\,\hat{i}\right]\cdot\left[\left(1+\cos\theta e^{-i\delta}\right)\hat{j} + \sin\theta e^{-i\delta}\,\hat{i}\right]$$

$$I_{avg} = \frac{E_0^2}{2\mu_0 c}\left[\left(1+\cos\theta e^{i\delta}\right)\left(1+\cos\theta e^{-i\delta}\right) + \sin^2\theta\right]$$

$$= \frac{E_0^2}{2\mu_0 c}\left(1+\cos^2\theta + \sin^2\theta + \cos\theta e^{i\delta} + \cos\theta e^{-i\delta}\right)$$

$$= \frac{E_0^2}{2\mu_0 c}\left[\ 2 + 2\cos\theta\left(\frac{e^{i\delta}+e^{-i\delta}}{2}\right)\right]$$

$$\boxed{I_{avg} = \frac{E_0^2}{\mu_0 c}\left[1+\cos\theta\cos\delta\right]}$$

Let's think on this. For $\theta = 0$, parallel sources, we get:

$I_{avg} = \frac{E_0^2}{\mu_0 c}\left[1+\cos\delta\right]$ reproducing our earlier answer.

For $\theta = \pi/2$, $\cos\theta = 0$, we get:

$I_{avg} = \frac{E_0^2}{\mu_0 c}$ which doesn't depend on the phase shift, as you'd expect it wouldn't

At this point you could throw in unequal frequencies or amplitudes, but really, the procedure stays the same.

The only notion that can qualitatively change some of this is the notion of <u>coherence</u>. But that is a different enough animal that we'll give it its own topic.