The following are some of the top questions of spring 2009 that are not addressed by the fall 2008 Q+A. In the previous Q+A I discuss the parameter L as well as the concept of half-range expansions. If you don't find what you need here then you might want to look at the previous Q+A.

## 1 Top Five Questions from Fourier Methods - Spring2009

1. What can Fourier series be used for outside of signals? What are real life applications? What does an engineer need to know Fourier series? If they are an approximation then how accurate are they? I'm a civil and I want to know how I will use this!

Since they can be used to represent 'any' periodic function, Fourier series can be used outside of signals. A good example of how they might get used would be to study periodic forcing of a structure. Suppose you have a mass–spring system with periodic force f(t),<sup>1</sup>

$$y'' + y = \sum_{n=1}^{\infty} a_n \cos\left((2n+1)t\right),$$
(1)

will this system resonate subject to this force? This is an important question for any engineered who will design structures prone to oscillation.<sup>2</sup> We will also see the Fourier series come back again when dealing with linear PDE defined on finite domains. They are a ubiquitous and prevalent theme in mathematics and physics and thus applied science. As for the error you may want to check out 11.6 of your textbook. We didn't get a chance to discuss error but we should remember that these methods tend to average so the error is reported as a mean square error and as with any discussion of error the first and most important question is how much error can you stand. The mathematics places clear controls on the error so you can calculate how many terms you would need to be within the error you want.

- 2. What does a Fourier transform show graphically? What does a Fourier transform tell us about the signal, which couldn't be seen otherwise? What is a Fourier transform? What are some applications? How does this fit into math? Does every function have a Fourier transform?
  - A Fourier transform depicts the intensity per each oscillatory frequency. From this perspective one can see the fundamental frequencies associated with the pre-transformed function. Good examples are the delta function examples of the last homework, which shows that if you want an even function with two fundamental modes of oscillation  $\omega_0$  and  $-\omega_0$  then this function is the cosine function. Maybe a clearer example is the buttons on a touch tone phone, when you press them it is unclear to the ear which you have pressed. However, if you take a Fourier transform of the tone then you will clearly see the fundamental frequencies of the tone and that these frequencies are different than if you press other buttons. This is how the 'machine' tells, which buttons you have pressed so that you can be connected to grandma who will send cookies.

<sup>&</sup>lt;sup>1</sup>One can think of a crystal as an array of mass-spring systems and more generally any structure, which can oscillate can be modeled crudely as mass-spring systems.

<sup>&</sup>lt;sup>2</sup>At the end of the day you might just want to damp the crap out of the system. http://en.wikipedia.org/wiki/Millennium\_ Bridge\_(London)

- There are too many applications of Fourier transforms to mention. However, I can say that anytime you have a phenomenon that has important frequency information they are very useful.<sup>3</sup> In the study of waves you might have a situation where waves are traveling along the surface of a body of water. Some of the waves might travel at different speeds. In fact, for certain equations one can show that waves of different wavelength (related to frequency) travel at different speeds. This is call dispersion and is fundamental to the study of many important nonlinear PDE.
- Not every function has a Fourier transform. However the class of functions, which posses Fourier transforms is quite large. There is much theory about this and a research area. People are constantly trying to apply these techniques in more general mathematical settings. In this way the Fourier methods appear in many parts of math both pure and applied. The most straightforward explanation of it all goes back to the statement that these methods split a function's information into intensities and oscillations and many phenomena we have fits into this description and can be better understood from this perspective.
- 3. How do you know how to periodically extend a function?
  - You don't. If given the choice you can do whatever you want b/c the function outside of the domain of definition doesn't care what you do. However as we will soon see the problem you are working on often chooses for you.
- 4. What is convolution? What is it used for?
  - Big question small answer. Convolution is the result of taking the Fourier transform of a product. That is to say the Fourier transform of a sum is the sum of Fourier transforms but the Fourier transform of a product is a so-called convolution integral. The physical interpretation is clearer from the perspective of linear differential equations, which says that convolution is a 'mixing' of two functions and that the solution to any forced linear differential equation can be written as the convolution of the Green's function (how the system responds to simple Dirac forcing) and the forcing function itself.
  - Convolution also appears in optics and probability. In optics one can show that the image on a photo plate is the convolution of the image itself and the lens geometry the light has to pass through. In probability one has the statement that the product of two probability density functions is the convolution of the two.
- 5. What the heck is a delta function?
  - The Dirac delta function is the idealization of a single impulse of information/energy/force given to a system. The idealization occurs when the duration of this impulse is reduced to an infinitesimal interval. In this way they are unrealistic but when this idealization occurs much of the calculus is simplified. They are also used in physics to consider point sources/particles/charges and though they seem sketchy the mathematics is rigorous but beside the point of usage.

<sup>&</sup>lt;sup>3</sup>They are also useful when there isn't any clear frequency information. They are just that awesome.

## 2 Quick Questions

- 1. Can a function be neither even nor odd? Yes!  $f(x) = \cos(x) + \sin(x)$ .
- 2. How did we connect matrices to Fourier series? Remember that from linear algebra we had the idea of linear combination and every matrix product could be thought of as a linear combination of vectors. Well, Fourier series are linear combinations of periodic functions used to make more periodic functions.
- 3. Where do the integral equations for the FS coefficients come from? They come from the orthogonality arguments of homework 5 problem 3.
- 4. Is there any way to deal with Gibb's phenomena? Yes, using the so-called Harr basis it is possible to mitigate this 'ringing.'
- 5. How does the Fourier integral relate to the Fourier transform? The Fourier integral can be used to derive the Fourier transform. See lecture 11 slides.
- 6. What is a Fourier mode? A Fourier mode is a single term of a Fourier series. The lowest order term is called the fundamental mode. Physically, the lowest mode tends to carry the most energy. This will be highlighted when we deal with a vibrating string. The lowest mode will contribute the most to the 'sound' while the other modes bring richness to the sound and are often called the higher order harmonics.
- 7. Are Fourier series more limited in usage than Fourier transforms since they require periodic functions? Fourier series is only a starting point for the field called harmonic analysis. The Fourier transform is the most general object we have and Fourier series can, in fact, be derived as a special case of Fourier transform through the use of the Dirac delta 'function.' The take away message is that the Fourier transform is the most general concept we have that decomposes a function into oscillatory modes and amplitudes of oscillation.
- 8. Is there a way to tell the periodicity of a Fourier series? Yes, look at the periodicity of the first term in the series. This dictates the periodicity of the Fourier series. If the first function is  $\cos(n\pi/2)$  then the function is 4-periodic.
- 9. How do you know when to use sine and cosine transforms? If the function you are transforming has an even/odd symmetry then the result of applying the complex Fourier transform is equivalent to application of cosine/sine transform.
- 10. What are cosine/sine series used for? Well, they come up if you have symmetry and maybe more importantly then are a consequence of the boundary conditions associated with PDE's.
- 11. Are there cases in which a Fourier series would be applied and a Fourier transform cannot be? No, a Fourier transform can reduce to a series.
- 12. If a function is not continuous then can a Fourier series show that discontinuity accurately? Yes, if the discontinuity happens to average the right hand and left hand limits going into the jump.
- 13. Where does the term Fourier come from? It's a guys name. He studied the flow of heat. http://en. wikipedia.org/wiki/Joseph\_Fourier

- 14. If sine is just a shifted cosine then can we just shift things and represent things in terms of one trig function? Yes, it is possible by the use of trig functions to introduce a phase shift and rewrite everything in terms of just one type of trig function. However, you would no longer have symmetry arguments. :(
- 15. How did people discover Fourier series? Though they were being studied in mathematical theory they also came up in the study of PDE's around the same time.
- 16. How can I learn more about Fourier methods? See me for more text books!
- 17. Are there other spaces to represent functions? Yes, there are many other transforms all of which give us a measure of the similarity of one function to another. There is a transform that looks like  $\int_0^\infty t^{-z} f(t) dt$ , which measures how similar f is to a power function. This wouldn't result in a frequency space but is still useful.
- 18. What is the point of the complex Fourier series? Nothing much, its the same as the real FS just fewer formulas.
- 19. What is the coolest thing a Fourier series/transform can do? Ton's of stuff. I guess the coolest thing I have done with them is shown the existence of solutions to quantum mechanical PDE that corresponds to a particle tunneling through a wall. Generally, Fourier methods give us a basis to understand many physical problems of common interest involving flows and vibrations.
- 20. What does the convergence or divergence of a Fourier series tell us? This is an indication of the 'space' where the function lives. Knowing about this can tell use energetic properties of the function.
- 21. How are Fourier series related to the topics of eigenvalue/eigenvectors of linear algebra? It turns out the the Fourier modes will be eigenvectors of the Laplacian operator, which we will encounter in PDE.
- 22. How do we determine if two functions are even when multiplied? See lecture 9 slides.
- 23. Is  $\hat{f}$  a vector, does it have a direction? No, this is a scalar function and the hat is used as a convention to know we are now in the Fourier domain.
- 24. Are coefficients always found the same way? Yes, but not by the same integrals. The coefficients were found b/c of orthogonality arguments. Other expansions have their own orthogonality conditions and thus their own coefficients. The integrals will differ though.
- 25. How do we tell if a function is odd or even? See lecture 9 slides.
- 26. Do Fourier methods ever fail? Yes, but not in anything you are likely to encounter in applied engineering.
- 27. How do we know when to use L's or  $\infty$ 's? Well, this depends on if you are using Fourier series or integrals/transforms.
- 28. If asked to find a Fourier series representation of a function where should I start? Start by writing down the coefficients and putting the function into them. Then integrate and write down the series.

- 29. How valid is this stuff for real time signal processing? Very valid. Though we are talking about theory the logic boards build to do this stuff have been optimized to do it quickly. It's like we have discussed the carnot cycle, which is far from what implemented under the hood of your average car.
- 30. What do Fourier series do again? Short answer see lectures 9-10.
- 31. Does a Fourier series model something that is not periodic? No, the series itself is periodic. The only way to mess this up is by taking  $L \to \infty$ .
- 32. Can you take the Fourier series of a constant function even though it is neither periodic nor on a finite domain? Yes, a constant function is a periodic function. It repeats itself all the time!
- 33. How does a Dirac function apply to the real world? It doesn't it is only an idealization. There is not such thing as a point particle but we use them all the time. The Dirac function is like a point particle.
- 34. How do you do FT on sets of data that don't have a best fit curve? The discrete Fourier transform is used here.
- 35. What do the negative n values represent in a complex Fourier series? Negatives mean go clockwise along the unit circle. Positives mean go counter clockwise.
- 36. What is the connection between Fourier series/integral/transform? See lecture 9 slides.
- 37. What's a good way to see if a function is even/odd? Graph the function and see lecture 9 slides.
- 38. What about the error in the truncated series? Well, we didn't get a chance to get into this but the text has a fine section on approximation by Fourier series 11.6.
- 39. Are there cases where you can get  $\hat{f}$  instead of f? Yes, it is common to report  $\hat{f}$  in problems and from this you can determine important properties of  $\hat{f}$ . For example given  $\hat{f}$  for a harmonic oscillator one can determine the resonant frequencies.