

Lecture: Eigenvalues and EigenvectorsModule: 07

Suggested Problem Set: Suggested Problems : {3, 5, 13, 14,16, 19, 21}

February 5, 2009

Quote of Lecture 7	
<b>George Carlin:</b> By and large, language is a tool for concealing the truth.	
	May 12, 1937 June 22, 2008

Okay, we know about  $\mathbf{Ax} = \mathbf{b}$ , or if we don't then we have some places to look. Now we concentrate on a special version of this equation where  $\mathbf{b} = \lambda\mathbf{x}$ ,  $\lambda \in \mathbb{C}$  and we say that,

$$(1) \quad \mathbf{Ax} = \lambda\mathbf{x},$$

is an eigenvalue-eigenvector problem for the square matrix  $\mathbf{A}_{n \times n}$ . Specifically,  $\mathbf{x}$  is called the eigenvector corresponding to the eigenvalue  $\lambda$ . If we think of  $\mathbf{A}$  as a linear transformation then  $\lambda$  is a measure of the transformation in the  $\mathbf{x}$ -direction. The set of all eigenvectors and their corresponding eigenvalues then provides yet another characterization of the transformation defined by  $\mathbf{A}$ .

Solving (1) is a two part process:

- Calculate the characteristic equation from,

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0,$$

and find  $\lambda$  by solving for the roots of the polynomial. These roots are often called the spectrum of  $\mathbf{A}$  and can be denoted at  $\sigma(\mathbf{A})$ .

- Determine a basis for the null space of,

$$(\mathbf{A} - \lambda\mathbf{I}),$$

by solving  $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ . The basis vectors are eigenvectors associated with the particular  $\lambda$  used to calculate them. Sometimes, the collection of all eigenvectors is called an eigenbasis for  $\mathbf{A}$ . I

If the eigenbasis of a matrix forms a basis for  $\mathbb{R}^n$  then many interesting properties can be deduced. If this occurs AND the matrix is self-adjoint,  $\mathbf{A}^H = \mathbf{A}$ , then one can show that the spectrum is purely real and that the eigenbasis forms an orthonormal basis for  $\mathbb{R}^n$ .<sup>1</sup>

#### Lecture Goals

- Understand how the concept of linear rescaling is related to eigenvalue-eigenvector problems.
- Use previous concepts of linear algebra to deduce a method for calculating eigenvalues and eigenvectors.

#### Lecture Objectives

- Derive auxiliary equations needed to calculate eigenvalues and eigenvectors.
- Summarize  $2 \times 2$  theory.
- Calculate the eigenbasis of various 'instructive' matrices.

<sup>1</sup>This concept underpins the theoretical measurements of quantum particles and will be important in the study of physical PDE.