Given
$$A = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix} \quad \text{for} \quad \frac{d\vec{Y}}{dt} = A\vec{Y}$$

a) Eigenvalues:

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} -3-\lambda & \sqrt{2} \\ \sqrt{2} & -2-\lambda \end{bmatrix}\right) =$$

$$= (-3-\lambda)(-2-\lambda) + -2 = \lambda^2 + 5\lambda + 4 = (\lambda+4)(\lambda+1) = 0$$

$$\Rightarrow \lambda_1 = -1$$
$$\lambda_2 = -4$$

b)

Case $\lambda_1 = -1$:

$(A - \lambda I)\vec{v} = \vec{0}$

$$\begin{bmatrix} -3+1 & \sqrt{2} \\ \sqrt{2} & -2+1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -2v_1 + \sqrt{2}v_2 = 0$$

let $v_1 = \sqrt{2}$, $v_2 = 2$ $\Rightarrow \vec{v}^{(1)} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$

Case $\lambda_2 = -4$:

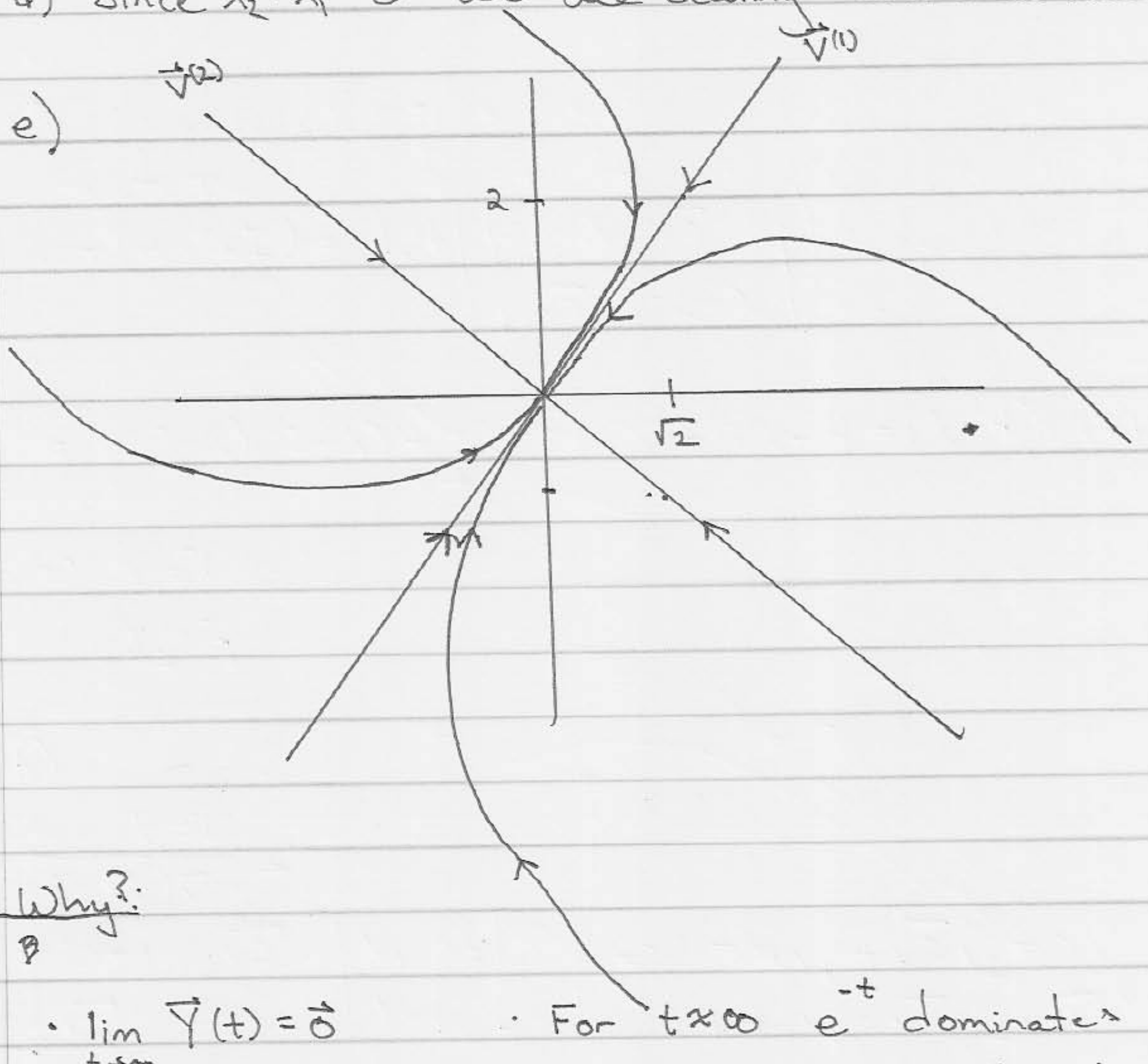
$$\begin{bmatrix} -3+4 & \sqrt{2} \\ \sqrt{2} & -2+4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_1 + \sqrt{2}v_2 = 0$$

let $v_1 = \sqrt{2}$, $v_2 = -1$ $\Rightarrow \vec{v}^{(2)} = \begin{bmatrix} \sqrt{2} \\ -1 \end{bmatrix}$

c) $\vec{Y}(t) = k_1 \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} e^{-t} + k_2 \begin{bmatrix} \sqrt{2} \\ -1 \end{bmatrix} e^{-4t}$

d) Since $\lambda_2 < \lambda_1 < 0$ we are dealing with a Real sink.

e)



Why?:

- $\lim_{t \to \infty} \vec{Y}(t) = \vec{0}$
- For $t \approx \infty$ e^{-t} dominates so soln are asymptotically like $\begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$
- For $t \approx -\infty$ e^{-4t} dominates so soln is asymptotically like $\begin{bmatrix} \sqrt{2} \\ -1 \end{bmatrix}$.

f) $\vec{Y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = k_1 \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} + k_2 \begin{bmatrix} \sqrt{2} \\ -1 \end{bmatrix} \Rightarrow 2k_1 = k_2$

$1 = \sqrt{2} k_1 + \sqrt{2} k_2 =$

$= \sqrt{2} k_1 + 2\sqrt{2} k_1 \Rightarrow k_1 = \frac{1}{3\sqrt{2}}$

$k_2 = \frac{2}{3\sqrt{2}}$

and

$\vec{Y}(t) = \frac{1}{3\sqrt{2}} \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} e^{-t} + \frac{2}{3\sqrt{2}} \begin{bmatrix} \sqrt{2} \\ -1 \end{bmatrix} e^{-4t}$

Given,

$$A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \quad \text{for} \quad \frac{d\vec{Y}}{dt} = A\vec{Y}$$

a) Eigenvalues:

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix}\right) = (1-\lambda)^2 - 4 =$$

$$= \lambda^2 - 2\lambda + 1 - 4 = \lambda^2 - 2\lambda - 3 = 0$$
$$(\lambda+1)(\lambda-3) = 0$$

$$\Rightarrow \quad \lambda_1 = -1$$
$$\lambda_2 = 3$$

b) Eigenvectors

Case $\lambda_1 = -1$:

$$(A - \lambda_1 I)\vec{v} = \begin{bmatrix} 1-(-1) & 1 \\ 4 & 1-(-1) \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} 2v_1 + v_2 = 0 \Rightarrow 2v_1 = -v_2 \\ 4v_1 + 2v_2 = 0 \end{array}$$

if we let $v_1 = 1$ then $v_2 = -2$ $\Rightarrow \vec{v}^{(1)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

* We can do this in $\mathbb{R}^2$ since any constant multiple of an Eigenvector is again an Eigenvector.

Case $\lambda_2 = 3$:

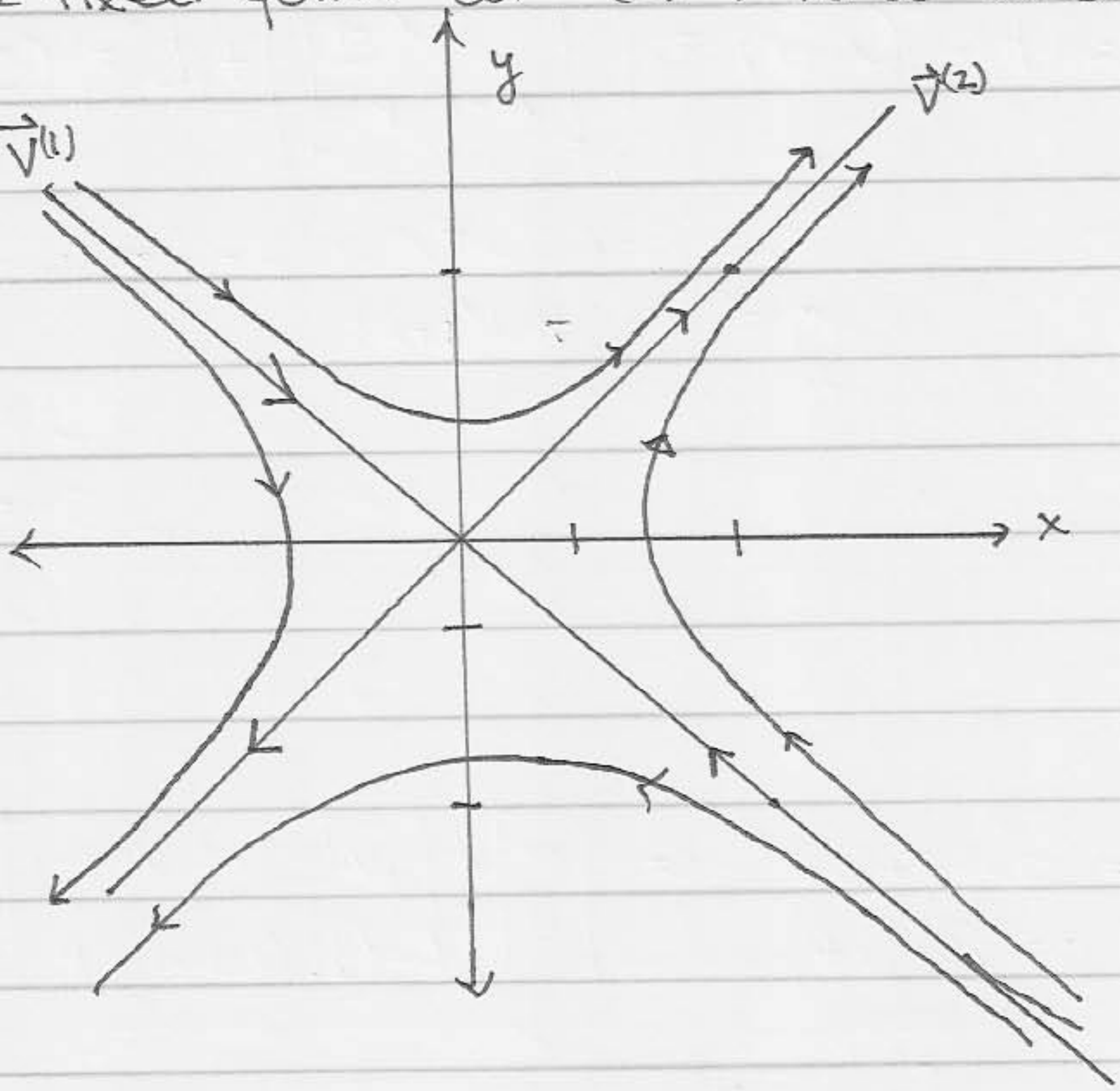
$$(A - \lambda I)\vec{v} = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \vec{v}^{(2)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

c) General Soln.

$$\vec{Y}(t) = k_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t} + k_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}, \; k_1, k_2 \in \mathbb{R}$$

d) Since $\lambda_1 = -1 < 0 < 3 = \lambda_2$ we conclude that the fixed point at $(0,0)$ is a saddle.

e)



Why:

- If the initial condition is such that $k_1 = 0$ then $e^{3t} \to$ growth along $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

- If the initial condition is such that $k_2 = 0$ then $e^{-t} \to$ decay along $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$
- If ~~t~~ $t \approx -\infty$ then $e^{3t} \approx 0$ and the soln is asymptotic to $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$. If $t \approx \infty$ then $e^{-t} \approx 0$ soln is asy. to $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$