FRESNEL'S FORMULAE

The geometry of reflected and transmitted waves has been obtained using only the wave character of light; however, nothing about the amplitudes of the waves has been determined. We must use Maxwell's equations and the boundary conditions associated with these equations to learn about the amplitudes of the reflected and transmitted waves. The geometry to be used in this discussion is shown in Figure 3-4. Two media are separated by an interface, the x, y plane at $z = 0$, whose normal $\hat{\mathbf{n}} = \hat{\mathbf{k}}$ is the unit vector along the z direction. The incident wave is labeled with an i, the reflected wave by an r, and the transmitted wave by a t. The incident wave's propagation vector $\mathbf{k}_i$, which we assume to lie in the x, z plane, and the normal to the interface establish the plane of incidence.

The electric field vectors for each of the three waves have been decomposed into two components: one in the plane of incidence, labeled P, and

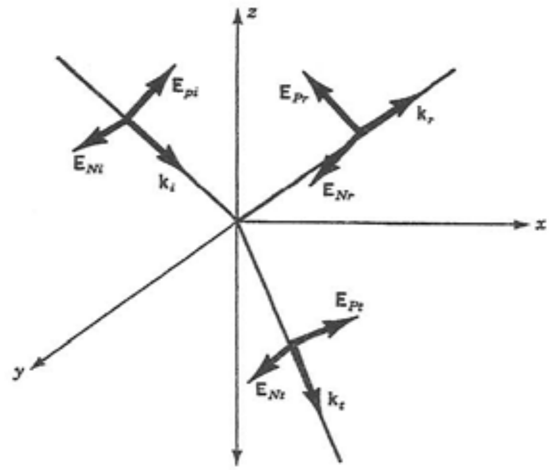

FIGURE 3-4. Orientation of the electric field and wave vectors in the coordinate system we selected for the discussion of reflection and refraction. The plane of incidence is the x, z plane.

one normal to the plane and parallel to the unit vector $\hat{\mathbf{j}}$ along the y axis, labeled N. This is an extension of the technique, discussed in Chapter 2, of using orthogonal vectors to describe the polarization of a light wave. (According to custom, the two polarizations are labeled π for parallel to the plane of incidence and σ for perpendicular to the plane of incidence. The Greek letter σ denotes perpendicular because *s* is the first letter of the German word *senkrecht*. We will use N and P in this book in place of the Greek letters.) The upper half-plane has a velocity of propagation v_i and an index of n_i and the lower half-plane has a velocity of propagation v_t and an index of n_t.

The actual vectors to be used are as follows:

Incident Wave

$$\mathbf{k}_i = k_i\,(\hat{\mathbf{i}} \sin\,\theta_i - \hat{\mathbf{k}} \cos\,\theta_i) \tag{3-11}$$

$$\mathbf{E}_i = E_{Pi}\,(\hat{\mathbf{i}} \cos\,\theta_i + \hat{\mathbf{k}} \sin\,\theta_i) + E_{Ni}\,\hat{\mathbf{j}} \tag{3-12}$$

Reflected Wave

$$\mathbf{k}_r = k_i\,(\hat{\mathbf{i}} \sin\,\theta_i + \hat{\mathbf{k}} \cos\,\theta_i) \tag{3-13}$$

$$\mathbf{E}_r = E_{Pr}\,(-\hat{\mathbf{i}} \cos\,\theta_i + \hat{\mathbf{k}} \sin\,\theta_i) + E_{Nr}\,\hat{\mathbf{j}} \tag{3-14}$$

Transmitted Wave

$$\mathbf{k}_t = k_t\,(\hat{\mathbf{i}} \sin\,\theta_t - \hat{\mathbf{k}} \cos\,\theta_t) \tag{3-15}$$

$$\mathbf{E}_t = E_{Pt}\,(\hat{\mathbf{i}} \cos\,\theta_t + \hat{\mathbf{k}} \sin\,\theta_t) + E_{Nt}\,\hat{\mathbf{j}} \tag{3-16}$$

The boundary conditions associated with Maxwell's equations are the following:

1. From $\nabla \cdot \mathbf{D} = \rho$, the normal components of $\mathbf{D}$ must be continuous if there are no surface charges. We use $\mathbf{D} = \epsilon\mathbf{E}$ to write this boundary condition

$$\left[\epsilon_i\,(\mathbf{E}_i + \mathbf{E}_r) - \epsilon_t\mathbf{E}_t\right]\cdot\hat{\mathbf{n}} = 0 \tag{3-17}$$

Evaluating the dot product yields

$$\epsilon_i \sin\,\theta_i\,(E_{Pi} + E_{Pr}) = \epsilon_t \sin\,\theta_t E_{Pt} \tag{3-18}$$

2. From the Maxwell's equation containing $\nabla \times \mathbf{E}$, we see that the tangential component of $\mathbf{E}$ is continuous. This boundary condition is written

$$(\mathbf{E}_i + \mathbf{E}_r + \mathbf{E}_t)\times\hat{\mathbf{n}} = 0 \tag{3-19}$$

Each one of the vector products is of the form

$$\mathbf{E}\times\hat{\mathbf{n}} = E_y\,\hat{\mathbf{i}} - E_x\,\hat{\mathbf{j}}$$

Evaluating the cross product yields

$$(E_{Ni} + E_{Nr} - E_{Nt})\,\hat{\mathbf{i}} - (E_{Pi}\cos\theta_i - E_{Pr}\cos\theta_i - E_{Pt}\cos\theta_t)\,\hat{\mathbf{j}} = 0$$

Both vector components must be equal to zero, thus,

$$E_{Ni} + E_{Nr} = E_{Nt} \tag{3-20}$$

$$(E_{Pi} - E_{Pr})\cos\theta_i = E_{Pt}\cos\theta_t \tag{3-21}$$

3. From $\nabla\cdot\mathbf{B} = 0$, the normal component of **B** must be continuous. We use **(2-17)** to rewrite the normal component of **B** in terms of **E**

$$\mathbf{B}\cdot\hat{\mathbf{n}} = \left(\frac{\sqrt{\mu\epsilon}}{k}\right)\ \mathbf{k}\times\mathbf{E}\cdot\hat{\mathbf{n}}$$

The boundary condition is then written

$$\left[\frac{\sqrt{\mu_i\epsilon_i}}{k_i}(\mathbf{k}_i\times\mathbf{E}_i + \mathbf{k}_r\times\mathbf{E}_r) - \frac{\sqrt{\mu_t\epsilon_t}}{k_t}(\mathbf{k}_t\times\mathbf{E}_t)\right]\cdot\hat{\mathbf{n}} = 0 \tag{3-22}$$

Each of the vectors will be of the form

$$(\mathbf{k}\times\mathbf{E})\cdot\hat{\mathbf{n}} = \Big[(E_yk_z - E_zk_y)\,\hat{\mathbf{i}} + (E_zk_x - E_xk_z)\,\hat{\mathbf{j}} + (E_xk_y - E_yk_x)\,\hat{\mathbf{k}}\Big]\cdot\hat{\mathbf{k}} = -E_yk_x$$

$$\sqrt{\mu_i\epsilon_i}\,(E_{Ni} + E_{Nr})\sin\theta_i = \sqrt{\mu_t\epsilon_t}\,E_{Nt}\sin\theta_t \tag{3-23}$$

We can simplify this relationship by rewriting Snell's law **(3-10)** as

$$\frac{\sin\theta_t}{\sin\theta_i} = \sqrt{\frac{\mu_i\epsilon_i}{\mu_t\epsilon_t}} \tag{3-24}$$

Equation **(3-23)** can then be seen to be the same as **(3-20)**. This boundary condition is redundant and we need not use it.

4. From the Maxwell's equation containing $\nabla\times\mathbf{H}$, we see that the tangential component of **H** is continuous if there are no surface currents. The tangent component of **H** can be written in terms of the electric field

$$\mathbf{H}\times\hat{\mathbf{n}} = \frac{\mathbf{B}}{\mu}\times\hat{\mathbf{n}} = \frac{\sqrt{\mu\epsilon}}{\mu k}\mathbf{k}\times\mathbf{E}\times\hat{\mathbf{n}} \tag{3-25}$$

The boundary condition is then written

$$\left[\frac{1}{k_i}\sqrt{\frac{\epsilon_i}{\mu_i}}(\mathbf{k}_i\times\mathbf{E}_i + \mathbf{k}_r\times\mathbf{E}_r) - \frac{1}{k_t}\sqrt{\frac{\epsilon_t}{\mu_t}}(\mathbf{k}_t\times\mathbf{E}_t)\right]\times\hat{\mathbf{n}} = 0$$

$$(\mathbf{k}\times\mathbf{E})\times\hat{\mathbf{n}} = \Big[(E_yk_z - E_zk_y)\,\hat{\mathbf{i}} + (E_zk_x - E_xk_z)\,\hat{\mathbf{j}} + (E_xk_y - E_yk_x)\,\hat{\mathbf{k}}\Big]\times\hat{\mathbf{k}}$$

$$= (E_zk_x - E_xk_z)\,\hat{\mathbf{i}} - E_yk_z\,\hat{\mathbf{j}} = 0 \tag{3-26}$$

Each vector component must equal zero. The x component gives

$$\sqrt{\frac{\epsilon_i}{\mu_i}}\left[E_{Pi} + E_{Pr}\right] = \sqrt{\frac{\epsilon_t}{\mu_t}}E_{Pt} \tag{3-27}$$

If we apply Snell's law **(3-24)** to **(3-18)**, we obtain the same equation as **(3-27)**. Therefore, we do not need boundary condition 1.

The y component of **(3-26)** gives

$$\sqrt{\frac{\epsilon_i}{\mu_i}}\,(E_{Ni} - E_{Nr})\cos\theta_i = \sqrt{\frac{\epsilon_t}{\mu_t}}\,E_{Nt}\cos\theta_t \tag{3-28}$$

Of the four boundary conditions of Maxwell's equations only two are needed to obtain the relationships between the incident, reflected, and transmitted waves; the conditions utilized are that the tangential components of **E** and **H** are continuous across the boundary. The boundary conditions place independent requirements on the polarizations parallel to and normal to the plane of incidence and generate two pair of equations that are treated separately. There are three unknowns but only two equations for each polarization; thus, the amplitudes of the reflected and transmitted light can only be found in terms of the incident amplitude.

σ Case (Perpendicular Polarization)

For this component of the polarization, **E** is perpendicular to the plane of incidence, that is, the x, z plane. This means that **E** is everywhere normal to $\hat{\mathbf{n}}$ and parallel to the boundary surface between the two media. [This case could be labeled the *transverse electric field* (TE) case; we do not use this notation because it implies that the wave may not be a transverse electromagnetic wave (a TEM wave); instead, we use the subscript N. The TE notation will be reserved for inhomogeneous waves in a guiding structure, discussed in Chapter 5.]

The second and fourth boundary conditions provide in **(3-20)** and **(3-28)** relationships between the various normal electric fields. The amplitude of reflected and transmitted light will be found, using these equations, in terms of the incident amplitude. We use Snell's law **(3-24)** to modify **(3-28)**

$$E_{Ni} - E_{Nr} = \frac{\mu_i \sin\theta_i}{\mu_t \cos\theta_i}\cdot\frac{\cos\theta_t}{\sin\theta_t}\cdot E_{Nt} = \frac{\mu_i \tan\theta_i}{\mu_t \tan\theta_t}E_{Nt} \tag{3-29}$$

Adding **(3-29)** to **(3-20)** yields

$$2E_{Ni} = \left(1 + \frac{\mu_i \tan\theta_i}{\mu_t \tan\theta_t}\right)E_{Nt}$$

$$\frac{E_{Nt}}{E_{Ni}} = \frac{2}{1 + \dfrac{\mu_i \tan\theta_i}{\mu_t \tan\theta_t}} \tag{3-30}$$

As was shown in Chapter 2, for the majority of optical materials, $\mu_i \approx \mu_t$ and the equation simplifies to

$$t_N = \frac{E_{Nt}}{E_{Ni}} = \frac{2\sin\theta_t\cos\theta_i}{\sin(\theta_i + \theta_t)} \tag{3-31}$$

The amplitude ratio t_N is called the *amplitude transmission coefficient for perpendicular polarization.*

Now **(3-30)** can be substituted back into **(3-20)** to obtain the amplitude ratio for the reflected light.

$$E_{Ni} + E_{Nr} = \frac{2E_{Ni}}{1 + \dfrac{\mu_i \tan\theta_i}{\mu_t \tan\theta_t}}$$

$$\frac{E_{Nr}}{E_{Ni}} = \frac{1 - \left(\dfrac{\mu_i \tan\theta_i}{\mu_t \tan\theta_t}\right)}{1 + \left(\dfrac{\mu_i \tan\theta_i}{\mu_t \tan\theta_t}\right)} \tag{3-32}$$

Again when $\mu_i \approx \mu_t$, we obtain the ratio of reflected amplitude to the incident amplitude, which is called the *amplitude reflection coefficient for perpendicular polarization.*

$$r_n = \frac{E_{Nr}}{E_{Ni}} = \frac{-\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)} \tag{3-33}$$

π Case (Parallel Polarization)

For this component of polarization, **E** is everywhere parallel to the plane of incidence; however, **B** and **H** are everywhere normal to $\hat{\mathbf{n}}$ and parallel to the boundary between the two media. [This case could be labeled the *transverse magnetic field* (TM) case, but we will use the subscript P for the same reason we did not use the TE notation. See Chapter 5, where the TM notation is utilized for inhomogeneous waves.] The second boundary condition provides **(3-21)** that can be written

$$E_{Pi} - E_{Pr} = \frac{\cos\theta_t}{\cos\theta_i} E_{Pt} \tag{3-34}$$

Applying Snell's law **(3-24)** to **(3-27)** yields

$$E_{Pi} + E_{Pr} = \sqrt{\frac{\mu_i \epsilon_t}{\mu_t \epsilon_i}}\, E_{Pt} = \frac{\mu_i \sin\theta_i}{\mu_t \sin\theta_t} E_{Pt} \tag{3-35}$$

Adding this equation to **(3-34)** yields the desired ratio of amplitudes

$$\frac{E_{Pt}}{E_{Pi}} = \frac{2\cos\theta_i \sin\theta_t}{\cos\theta_t \sin\theta_t + \dfrac{\mu_i}{\mu_t}\cos\theta_i \sin\theta_i} \tag{3-36}$$

For the usual situation of $\mu_i \approx \mu_t$, the *amplitude transmission coefficient for parallel polarization* is

$$t_P = \frac{E_{Pt}}{E_{Pi}} = \frac{2\cos\theta_i \sin\theta_t}{\sin(\theta_i + \theta_t)\cos(\theta_i - \theta_t)} \tag{3-37}$$

Substituting **(3-37)** into **(3-35)** produces the amplitude reflection ratio

$$\frac{E_{Pr}}{E_{Pi}} = \frac{\left(\dfrac{\mu_i}{\mu_t}\right)\sin 2\theta_i - \sin 2\theta_t}{\sin 2\theta_t + \left(\dfrac{\mu_i}{\mu_t}\right)\sin 2\theta_i} \tag{3-38}$$

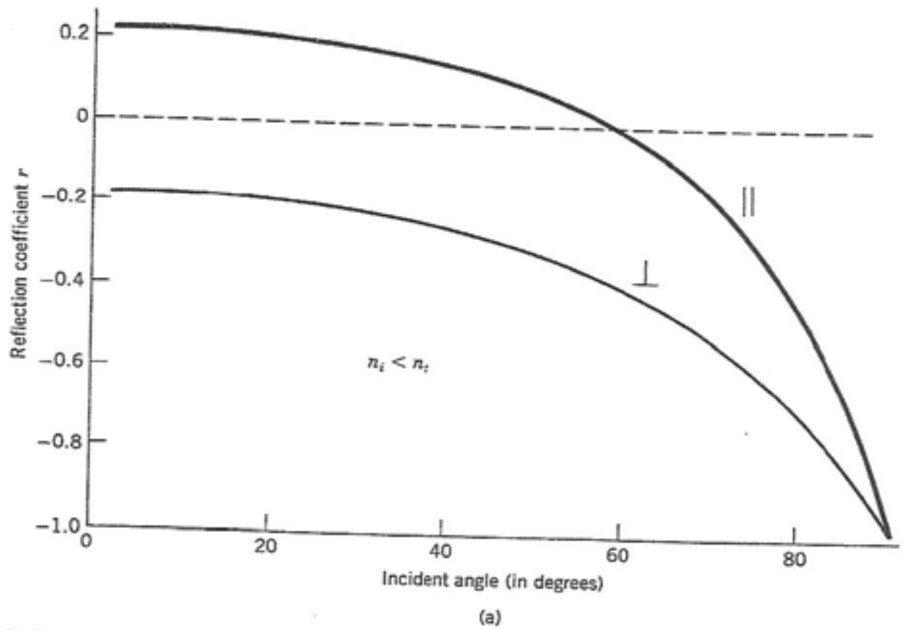

FIGURE 3-5a. Reflection coefficient for $n_i = 1.0$ and $n_t = 1.5$, i.e., the ratio of index of refraction is 1.5.

The assumption of $\mu_i \approx \mu_t$ (from now on this assumption will be used) results in the *amplitude reflection coefficient for parallel polarization* of

$$r_P = \frac{E_{Pr}}{E_{Pi}} = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)} \tag{3-39}$$

The amplitude reflection coefficients are plotted as a function of the incident angle in Figure 3-5. Figure 3-5*a* corresponds to the condition of

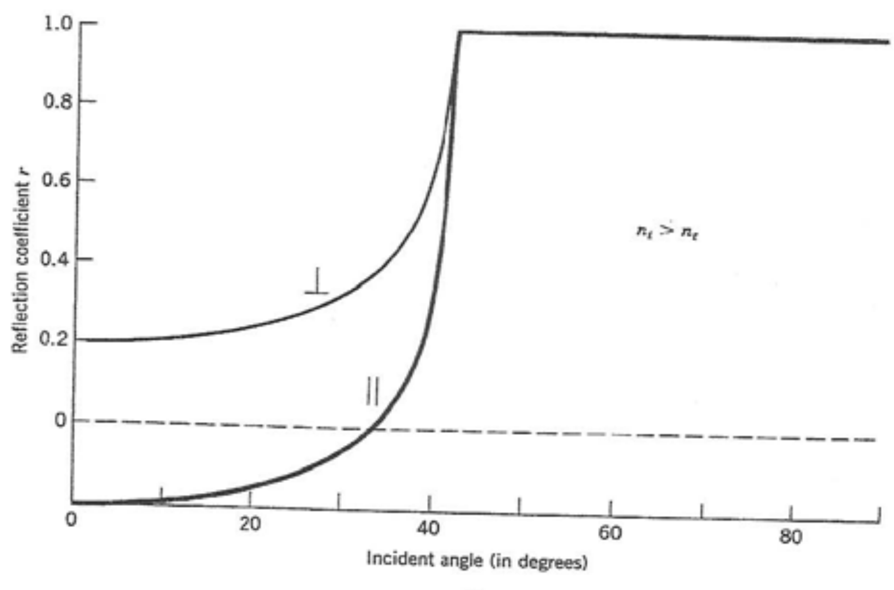

FIGURE 3-5b. Reflection coefficient for $n_i = 1.5$ and $n_t = 1$, i.e., ratio of index of refraction is 0.67.

$n_i < n_t$ and Figure 3-5*b* corresponds to the condition of $n_i > n_t$. As can be seen by examining the two figures, the behavior of the reflection coefficient is quite different for these two conditions.

The plots of the reflection coefficients shown in Figure 3-5*a* and 3-5*b* demonstrate that a sign change occurs for r_P, labeled $\|$ in the figure, for a range of angles that depend on the relative index of refraction. This phase change is important because r_P must pass through zero for the phase change to occur.

We will discuss in detail the behavior of the amplitude reflection coefficient when $\theta \approx 0°$, $r_P = 0$, and $r = 1$, which occurs if $n_i > n_t$, as shown in Figure 3-5*b*.

REFLECTED AND TRANSMITTED ENERGY

The fraction of the incident amplitude reflected and transmitted at a surface is not experimentally available. The parameter that can be measured is the energy. At first, we might think that we could simply square the ratios we have derived to obtain the energies but this would lead to erroneous results. The correct way to proceed is to use the average Poynting vector incident on a unit area, given by

$$\langle \mathbf{S} \rangle \cdot \hat{\mathbf{n}} = |\langle \mathbf{S} \rangle| \cos\,\theta_i$$

where the expression for the average Poynting vector is obtained from **(2-26)**. (We assume that $\mu = \mu_0$, resulting in

$$\mu c = \sqrt{\frac{\mu_0}{\epsilon_0}}$$

which is defined as the impedance of vacuum.) The energy flow across the boundary can be obtained from the following equations:

$$v_i \langle U_i \rangle = \langle S_i \rangle \cos\,\theta_i = \frac{n_i}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}\,|E_i|^2 \cos\,\theta_i \tag{3-40}$$

$$v_i \langle U_r \rangle = \langle S_r \rangle \cos\,\theta_i = \frac{n_i}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}\,|E_r|^2 \cos\,\theta_i \tag{3-41}$$

$$v_t \langle U_t \rangle = \langle S_t \rangle \cos\,\theta_t = \frac{n_t}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}\,|E_t|^2 \cos\,\theta_t \tag{3-42}$$

Each of these three equations apply separately to the normal and parallel components of polarization, resulting in six equations for the description of reflected and transmitted energy at a boundary.

We define *reflectivity* as

$$R = \frac{\langle U_r \rangle}{\langle U_i \rangle} = \frac{|E_r|^2}{|E_i|^2} \tag{3-43}$$

and *transmissivity* as

$$T = \frac{v_t \langle U_t \rangle}{v_i \langle U_i \rangle} = \left(\frac{n_t}{n_i}\right)\frac{\cos\,\theta_t}{\cos\,\theta_i}\cdot\frac{|E_t|^2}{|E_i|^2} \tag{3-44}$$

[The quantities defined by **(3-43)** and **(3-44)** are ratios of the Poynting vectors and therefore assume a wave of known frequency and phase. Experimentally, the ratios of the incident flux to the reflected and transmit-

Appendix 4-A

MULTILAYER DIELECTRIC COATINGS

Interference between waves reflected from the interfaces of a dielectric film can be used to reduce the reflection from an optical surface. This concept can be extended to a multilayer dielectric coating to produce any desired reflection property, and in this appendix, several procedures for designing multilayer coatings will be described.

Fraunhofer produced antireflection layers on glass surfaces by acid etching in 1817 but it was not until 1891 that **Dennis Taylor** associated the reduced reflectivity with an increased transparency. Full utilization of the concept of interference filters had to wait until the late 1940s and early 1950s when techniques were developed that allowed the production of rugged dielectric films. Cold mirrors designed to reflect the visible wavelengths and transmit the infrared wavelengths were one of the first products of this technology and today are found in every dentist's lamp. In the 1970s the multilayer coating technology had developed to a point that allowed the mass production of laser mirrors with very low absorption. Coating technology is now being used to produce durable mirrors for copy machines and conductive coatings to provide frost-free aircraft windshields.

We have introduced the theory of multilayer interference filters in Chapters 3 and 4. This theory will now be used to develop several methods for designing a filter. Only the reflectivity of the filter will be considered, but enough information will be given to allow transmission to also be calculated. We need a notation that will allow the manipulation of a large number of dielectric layers. To see how the notation is generated, we rewrite **(3-33)** as

$$r_N = \frac{E_{Nr}}{E_{Ni}} = \frac{n_i \cos \theta_i - n_t \cos \theta_t}{n_i \cos \theta_i + n_t \cos \theta_t}$$

and **(3-39)** as

$$r_P = \frac{E_{Pr}}{E_{Pi}} = \frac{\dfrac{n_i}{\cos \theta_i} - \dfrac{n_t}{\cos \theta_t}}{\dfrac{n_i}{\cos \theta_i} + \dfrac{n_t}{\cos \theta_t}}$$

First, we defined the ith medium as the incident medium and the $(i + 1)$ medium as the transmitted medium. The two reflection coefficients can be written as one coefficient by defining an effective index. The effective index for normal polarization is

$$N_j = \mathcal{N}_j \cos \theta_j$$

and for parallel polarization is

$$N_j = \frac{\mathcal{N}_j}{\cos\theta_j}$$

where the notation indicates that the dielectric films can have a complex index of refraction. This notation allows generalized equations for the amplitude reflection coefficient and transmission coefficient to be defined at the interface between medium i and $i + 1$

$$r_i = \frac{N_i - N_{i+1}}{N_i + N_{i+1}}, \qquad r'_i = -r_i = \frac{N_{i+1} - N_i}{N_i + N_{i+1}} \qquad \text{(4A-1a)}$$

$$t_i = \frac{2N_i}{N_i + N_{i+1}}, \qquad t'_i = \frac{2N_{i+1}}{N_i + N_{i+1}} \qquad \text{(4A-1b)}$$

[When $\theta_j = 0$, the generalized equation **(4A-1)** agrees with **(3-45)** and **(3-46)**.] The subscripts allow an unlimited number of interfaces. We will start at $j = 0$ and for m layers there will be $m + 2$ indices of refraction n_0, $n_1, \ldots, n_m, n_{m+1}$, where n_0 is the refractive index in the incident medium and n_{m+1} is the refractive index of the substrate medium, that is, the medium on which the dielectric layers will be deposited. This notation has allowed the design and construction of multilayer stacks as large as $m > 100$.

TABLE 4A.1 Thin-Film Materials

Material	Index of Refraction	Wavelength Range, μm
Cryolite(Na_3AlF_6)	1.35	0.15–14
Magnesium fluoride (MgF_2)	1.38	0.12–8
Silicon dioxide (SiO_2)	1.46	0.17–8
Thorium fluoride (ThF_4)	1.52	0.15–13
Aluminum oxide (Al_2O_3)	1.62	0.15–6
Silicon monoxide (SiO)	1.9	0.5–8
Zirconium dioxide (ZrO_2)	2.00	0.3–7
Cerium dioxide (CeO_2)	2.2	0.4–16
Titanium dioxide (TiO_2)	2.3	0.4–12
Zinc sulfide (ZnS)	2.3	0.4–12
Zinc selenide (ZnSe)	2.44	0.5–20
Cadmium telluride (CdTe)	2.69	1.0–30
Silicon (Si)	3.5	1.1–10
Germanium (Ge)	4.05	1.5–20
Lead telluride (PbTe)	5.1	3.9–20+

Matrix Approach

A quantitative approach to designing multilayer filters that allows the computer generation of the spectral characteristics of the filters is the use of matrices. The approach is called the *method of resultant waves* or the E^+, E^- *matrix method*.[16] The boundary conditions associated with Maxwell's equations, as stated in Chapter 4, are placed into a matrix equation format. To accomplish the reformulation, the boundary conditions are manipulated so that the information about the angle of incidence and the polarization are

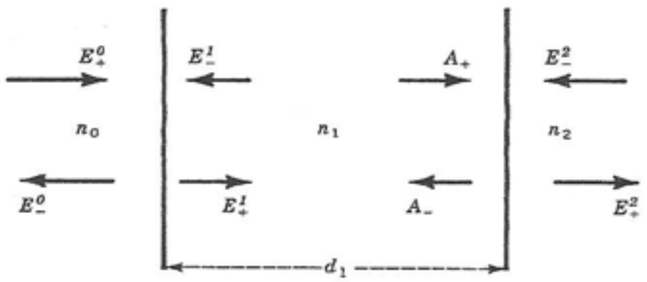

FIGURE 4A-5. Geometry for waves in a dielectric film.

placed into an effective index of refraction as defined in **(4A-1)**. The fields on each side of the boundary can then be represented by plane waves incident normal to the interface as shown in Figure 4A-5.

Each dielectric layer has two interfaces but the two interfaces, as shown in Figure 4A-5, are formally identical so we need only configure the problem for a general interface and repeat the calculation m times for the m interfaces of a dielectric stack $m - 1$ layers high.

We use the boundary conditions **(3-20, 3-21, 3-27, 3-28)** from Chapter 3 to produce the boundary conditions at a generalized interface.

$$n_i \cos \theta_i (E^i_{N+} - E^i_{N-}) = n_{i+1} \cos \theta_{i+1} (E^{i+1}_{N+} - E^{i+1}_{N-}) \qquad (4A\text{-}4)$$

$$E^i_{N+} + E^i_{N-} = E^{i+1}_{N+} + E^{i+1}_{N-} \qquad (4A\text{-}5)$$

$$(E^i_{P+} - E^i_{P-}) \cos \theta_i = (E^{i+1}_{P+} - E^{i+1}_{P-}) \cos \theta_{i+1} \qquad (4A\text{-}6)$$

$$(E^i_{P+} + E^i_{P-}) n_i = (E^{i+1}_{P+} + E^{i+1}_{P-}) \, n_{i+1} \qquad (4A\text{-}7)$$

For the normal components,

$$E^i_{N+} = \left(\frac{n_i \cos \theta_i + n_{i+1} \cos \theta_{i+1}}{2n_i \cos \theta_i} \right) E^{i+1}_{N+} + \left(\frac{n_i \cos \theta_i - n_{i+1} \cos \theta_{i+1}}{2n_i \cos \theta_i} \right) E^{i+1}_{N-} \qquad (4A\text{-}8)$$

$$E^i_{N-} = \left(\frac{n_i \cos \theta_i - n_{i+1} \cos \theta_{i+1}}{2n_i \cos \theta_i} \right) E^{i+1}_{N+} + \left(\frac{n_i \cos \theta_i + n_{i+1} \cos \theta_{i+1}}{2n_i \cos \theta_i} \right) E^{i+1}_{N-} \qquad (4A\text{-}9)$$

For the parallel components,

$$E^i_{P+} = \left(\frac{n_i \cos \theta_{i+1} + n_{i+1} \cos \theta_i}{2n_i \cos \theta_i} \right) E^{i+1}_{P+} + \left(\frac{n_{i+1} \cos \theta_i - n_i \cos \theta_{i+1}}{2n_i \cos \theta_i} \right) E^{i+1}_{P-} \qquad (4A\text{-}10)$$

$$E^i_{P-} = \left(\frac{n_{i+1} \cos \theta_i - n_i \cos \theta_{i+1}}{2n_i \cos \theta_i} \right) E^{i+1}_{P+} + \left(\frac{n_i \cos \theta_{i+1} + n_{i+1} \cos \theta_i}{2n_i \cos \theta_i} \right) E^{i+1}_{P-} \qquad (4A\text{-}11)$$

Using the definitions in **(4A-1a)** and **(4A-1b)**, we can simplify our notation and reduce **(4A-9)** through **(4a-12)** to the following two equations:

$$E_{+}^{i} = \frac{E_{+}^{i+1} + r_i E_{-}^{i+1}}{t_i} \tag{4A-12}$$

$$E_{-}^{i} = \frac{r_i E_{+}^{i+1} + E_{-}^{i+1}}{t_i} \tag{4A-13}$$

We may combine **(4A-12)** and **(4A-13)** into a matrix equation for the interface

$$\begin{pmatrix} E_{+}^{i} \\ E_{-}^{i} \end{pmatrix} = \begin{pmatrix} \frac{1}{t_i} & \frac{r_i}{t_i} \\ \frac{r_i}{t_i} & \frac{1}{t_i} \end{pmatrix} \begin{pmatrix} E_{+}^{i+1} \\ E_{-}^{i+1} \end{pmatrix}$$

Normally, this is written in a more compact notation

$$\boldsymbol{E}^{i} = \boldsymbol{I}_i \boldsymbol{E}^{i+1}$$

where $\boldsymbol{I}_i$ is the ith interface matrix

$$\boldsymbol{I}_i = \begin{pmatrix} \frac{1}{t_i} & \frac{r_i}{t_i} \\ \frac{r_i}{t_i} & \frac{1}{t_i} \end{pmatrix} \tag{4A-14}$$

The problem of finding the values of A_+ and A_- in Figure 4A-5 is a simple propagation problem. The fields A_+ and E_- must be modified by the phase shift they experience after propagating through the dielectric layer, here labeled 1

$$A_+ = e^{i\delta_1} E_+^1, \qquad E_-^1 = e^{i\delta_1} A_-, \qquad \text{or} \qquad A_- = e^{-i\delta_1} E_-^1$$

These equations can be combined into a matrix equation

$$\boldsymbol{A} = \boldsymbol{T}_1 \boldsymbol{E}^1$$

which can be generalized for the ith dielectric layer by defining a transmission matrix of the form

$$\boldsymbol{T}_i = \begin{pmatrix} e^{i\delta_i} & 0 \\ 0 & e^{-i\delta_i} \end{pmatrix} \tag{4A-15}$$

The effect of an m layer dielectric film can be described by the matrix equation

$$\begin{pmatrix} E_{+}^{0} \\ E_{-}^{0} \end{pmatrix} = \boldsymbol{M} \begin{pmatrix} E_{+}^{f} \\ E_{-}^{f} \end{pmatrix} \tag{4A-16}$$

where

$$\boldsymbol{M} = \boldsymbol{I}_0 \cdot \boldsymbol{T}_1 \cdot \boldsymbol{I}_1 \cdot \boldsymbol{T}_2 \cdot \cdots \cdot \boldsymbol{I}_{m-1} \cdot \boldsymbol{T}_m \cdot \boldsymbol{I}_m$$

and the E^f are the fields in the final medium. The reflection coefficient of the stack is

$$\rho = \frac{E_{-}^{0}}{E_{+}^{0}} \tag{4A-17}$$

To simplify the problem without affecting the desired ratios, we normally assume $E_+^f = 1$ and $E_-^f = 0$.

It is simple to program a personal computer to explore the properties of various combinations of films. We will sketch out a very simple example. The objective is to design a multilayer stack made up of alternating layers of index n_1 and n_2 with the light incident normal to the stack. The thickness of each layer is selected to be one-quarter wavelength thick at the design wavelength. A schematic representation of the film stack would have the following index layers:

$$n_0 \mid n_1 \mid n_2 \mid n_1 \mid \ldots \mid n_1 \mid n_g$$

The first step in the solution of the problem is to form three matrices. The front layer matrix is

$$\boldsymbol{F} = \boldsymbol{I}_0\boldsymbol{T}_1 = \frac{i}{2}\begin{bmatrix} 1+n_1 & -(1-n_1) \\ 1-n_1 & -(1+n_1) \end{bmatrix}$$

The back layer matrix is

$$\boldsymbol{B} = \boldsymbol{T}_{N+1}\boldsymbol{I}_{N+1} = \frac{i}{2n_1}\begin{bmatrix} n_1+n_g & n_1-n_g \\ -(n_1-n_g) & -(n_1+n_g) \end{bmatrix}$$

There are N pairs of layers of index n_1 and n_2 and the matrix for one pair is

$$\boldsymbol{M} = \boldsymbol{I}_1\boldsymbol{T}_2\boldsymbol{I}_2\boldsymbol{T}_1 = -\frac{1}{2}\begin{pmatrix} \dfrac{n_1}{n_2}+\dfrac{n_2}{n_1} & \dfrac{n_1}{n_2}-\dfrac{n_2}{n_1} \\ \dfrac{n_1}{n_2}-\dfrac{n_2}{n_1} & \dfrac{n_1}{n_2}+\dfrac{n_2}{n_1} \end{pmatrix}$$

For N pair of the $n_1 \mid n_2$ layers,

$$\boldsymbol{M}^N = \frac{(-1)^N}{2}\begin{pmatrix} \left(\dfrac{n_1}{n_2}\right)^N+\left(\dfrac{n_2}{n_1}\right)^N & \left(\dfrac{n_1}{n_2}\right)^N-\left(\dfrac{n_2}{n_1}\right)^N \\ \left(\dfrac{n_1}{n_2}\right)^N-\left(\dfrac{n_2}{n_1}\right)^N & \left(\dfrac{n_1}{n_2}\right)^N+\left(\dfrac{n_2}{n_1}\right)^N \end{pmatrix}$$

We will assume that $n_g = 1$, which allows us to write the transmission **(4A-15)** for this problem as

$$\begin{pmatrix} E_+^0 \\ E_-^0 \end{pmatrix} = \frac{(-1)^{N+1}}{8}\begin{pmatrix} \left(\dfrac{n_1}{n_2}\right)^N+\left(\dfrac{n_2}{n_1}\right)^N & \left(\dfrac{n_1}{n_2}\right)^N-\left(\dfrac{n_2}{n_1}\right)^N \\ \left(\dfrac{n_1}{n_2}\right)^N-\left(\dfrac{n_2}{n_1}\right)^N & \left(\dfrac{n_1}{n_2}\right)^N+\left(\dfrac{n_2}{n_1}\right)^N \end{pmatrix}\begin{pmatrix} E_+^f \\ E_-^f \end{pmatrix}$$

We can now write the reflection coefficient of the stack as

$$\rho = \frac{\left(\dfrac{n_1}{n_2}\right)^N-\left(\dfrac{n_2}{n_1}\right)^N}{\left(\dfrac{n_1}{n_2}\right)^N+\left(\dfrac{n_2}{n_1}\right)^N} \qquad \text{(4A-18)}$$

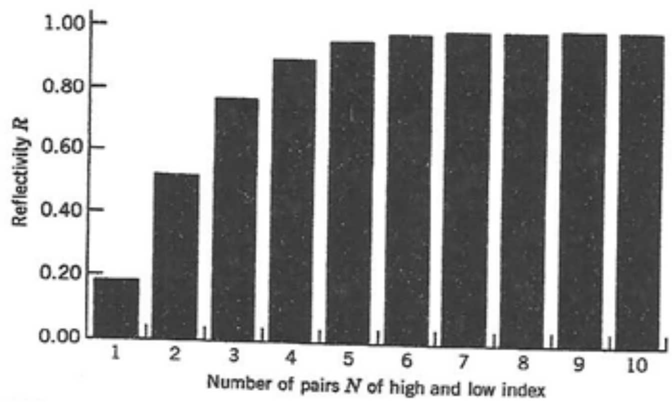

FIGURE 4A-6. The reflectivity as a function of the number of pairs of low-index and high-index films N is shown for a stack with a low-index film of 1.46 and a high-index film of 2.3.

We know that one of the indices is larger than the other. For large N, one of the two ratios will dominate and the reflectivity will approach 1, as shown in Figure 4A-6 where the reflectivities for $N = 1$ to 10 are displayed. As the difference between n_1 and n_2 increases, so does the wavelength range over which the reflectivity can be made nearly 1. A simple computer model using **(4A-18)** generated the curve in Figure 4A-7, which demonstrates the dependence of the reflectivity on the index ratio.

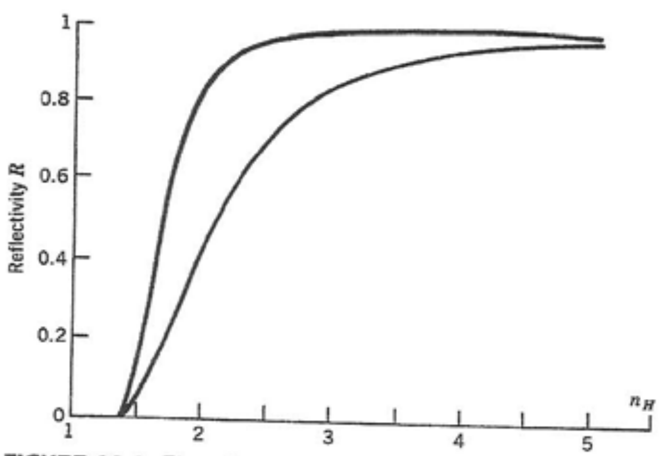

FIGURE 4A-7. The reflectivity of a dielectric stack as a function of the ratio of the index of refraction of the two materials used to construct the stack. The low index of refraction layer was set at 1.35 and the high index of refraction layer was allowed to range from 1.38 to 5.1. Two different size stacks were used, one with $N = 2$ and the other with $N = 4$.