Rectangular metal waveguides

solve for longitudinal field

TE: solve for B_z with $\partial B_z/\partial \hat{n} = 0$

i.e. zero slope at walls

TM: solve for E_z with $E_z = 0$ at walls.

extend 1-D solutions -

remember separation of variables → product of sol'ns

$$TE: \quad B_z(x,y,z) = B_{z0} \cos\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) e^{i(k_g^{(n,m)} z - \omega t)}$$

modes identified by index

TE_{mn}

$\underline{\text{propagation constant} \quad k_g^{(m,n)}} = k_z$

$$k_z = \sqrt{k_0^2 - k_x^2 - k_y^2}$$

$$= \left(\frac{\omega^2}{c^2} - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2\right)^{1/2}$$

if $a > b$, lowest mode (largest k_z) is TE_{10}

($TE_{00} = TEM_{00}$ can't propagate)

> cutoff for any mode is where $k_z = 0$

as freq. is tuned, reach diff't cutoffs:

$$\frac{\omega_{mn}^2}{c^2} = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \rightarrow \omega_{mn} = \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$$

> single-mode: if $\omega_{10} < \omega < \omega_{01}$

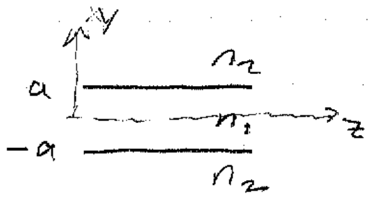
this range is single-mode bandwidth

Dielectric waveguides

qualitative differences

- loss-free guiding by TIR
- evanescent wave exists in cladding
  must match functions at boundaries.
- used in IR, visible, UV part of spectrum
- lowest order mode of symmetric wg. is not cut off.
- possible to have gradient index

EM solution for modes. one-D slab



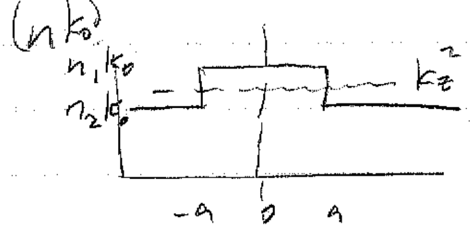
$$\frac{\partial^2 \vec{E}_T}{\partial y^2} + \varepsilon(y) k_0^2 \vec{E}_T = k_z^2 \vec{E}_T$$

$\varepsilon(x) = n^2(y)$

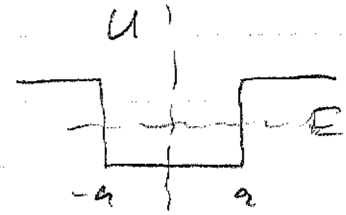
qualitative solution:

QM equivalent pot'l $\quad U(y) \sim -n^2(y) k_0^2$

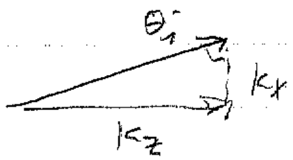
" eigenval $\quad E \sim -k_z^2$



QM $\rightsquigarrow$



- k_z is bounded by $n_2 k_0$ to $n_1 k_0$ for guided mode.
- lowest modes: $k_z \lesssim n_1 k_0$
  cutoff $\quad k_z \approx n_2 k_0$
- use ∞ well type solutions to estimate # modes



$n_1 \sin\theta_c = n_2$

$n_1 k_z / n_1 k_0 = n_2$

At cutoff (dielectric case) $k_z = n_2 k_0$
i.e. in cladding $k_y = 0$, evanescent field isn't damped

> estimate # bound modes using metal w.g. dispersion
inside w.g. (metal)

$$k_y = \frac{m\pi}{(2a)} \rightarrow k_z^2 = n_1^2 k_0^2 - \frac{m^2\pi^2}{4a^2} \geq n_2^2 k_0^2 \quad \text{at cutoff}$$

solve for m: bound if inequal. is true.

$$m^2 \leq (n_1^2 - n_2^2) k_0^2 a^2 \frac{4}{\pi^2}$$

V-number $V \equiv (n_1^2 - n_2^2)^{1/2} k_0 a$

mode m is bound if

$$m < V \cdot 2/\pi \quad \text{(estimate)}$$

estimate tends to be low by 1 or 2 modes.

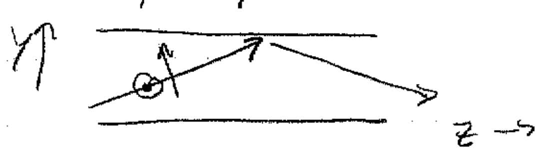
$V \approx$ # bound modes.

Solve for modes

$$\frac{\partial^2 \vec{E}}{\partial y^2} = -(n_i^2 k_0^2 - k_z^2)\vec{E} \qquad i = 1, 2$$

here, we'll solve for field that is purely transverse

TE: solve for E_x
TM solve for B_x

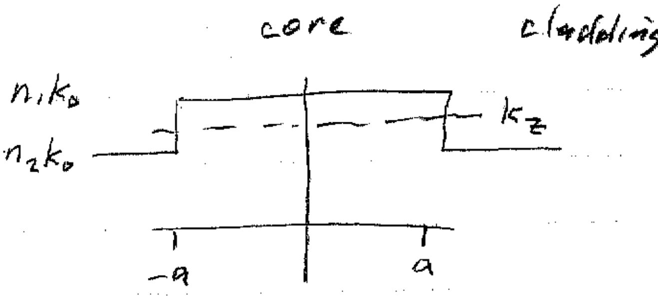


most cases,

$$\frac{n_1 - n_2}{n_1} \ll 1 \qquad \text{"weak-guiding"}$$

ignore variation in ε so that $\nabla \cdot (\varepsilon \vec{E}) \approx \varepsilon \nabla \cdot \vec{E}$

→ modes are close to TEM



Note k_y's depend on region.

for $|y| < a$

k_y = real, oscill. solns

let $k_y = \alpha$ in core $\quad E \sim \cos\alpha y$ or $\sin\alpha y$

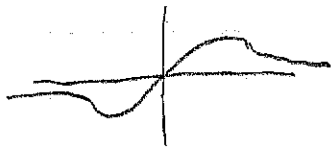
for $|y| > a$

k_y = imag (evanesc. wave)

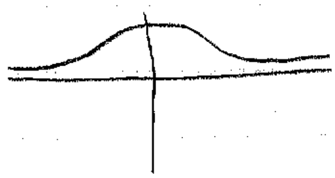
let $k_y = i\beta$ in cladding. $E \sim e^{\pm\beta x}$

Take advantage of symmetry:

odd $\to$ $\sin\alpha y$



even $\to$ $\cos\alpha y$



match at one boundary.

TE even solns

$$E_x = E_1 \cos\alpha y \qquad \text{core}$$
$$= E_2 e^{-\beta y} \qquad y > a \quad \text{cladding}$$

at $y = a$ match tang. compon.

$$E_1 \cos\alpha a = E_2 e^{-\beta a}$$

second B.C. from $\vec{B}$ $\quad \mu = 1$ everywhere, all $\vec{B}$ contin.

$$\nabla \times \vec{E} = i\frac{\omega}{c}\vec{B}$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ E_x & 0 & 0 \end{vmatrix} = \hat{y}\,\partial_z E_x - \hat{z}\,\partial_y E_x = i\frac{\omega}{c}\vec{B}$$

$\Delta B_z = 0 \Rightarrow \Delta(\partial_y E_x) = 0 \quad \therefore$ match slope.

$$E_x'\big|_a = -\alpha E_1 \sin \alpha a \qquad \text{core}$$

$$= -\beta E_2 e^{-\beta a} \qquad \text{cladding}$$

divide two eqns, eliminate E_1, E_2

$$\underline{\alpha a \tan \alpha a = \beta a}$$

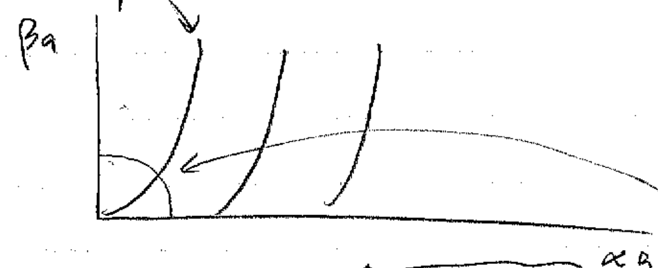
odd solution $\rightarrow$ $\underline{\alpha a \cot \alpha a = -\beta a}$

ultimately k_z is eigenvalue.

$$\alpha^2 = n_1^2 k_0^2 - k_z^2$$
$$\beta^2 = -(n_2^2 k_0^2 - k_z^2)$$

note $\alpha^2 + \beta^2 = n_1^2 k_0^2 - n_2^2 k_0^2$
$= V^2/a^2$

plot βa as fcn of αa



also plot $\beta a = \sqrt{V^2 - (\alpha a)^2}$ circles

intersect $\rightarrow$ modes.

numerically find roots.