Maxwell's Equations to wave eqn

- The induced polarization, **P**, contains the effect of the medium:

$$\vec{\nabla}\cdot\mathbf{E}=0 \qquad \vec{\nabla}\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}$$

$$\vec{\nabla}\cdot\mathbf{B}=0 \qquad \vec{\nabla}\times\mathbf{B}=\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}+\mu_0\frac{\partial\mathbf{P}}{\partial t}$$

Take the curl:

$$\vec{\nabla}\times\left(\vec{\nabla}\times\mathbf{E}\right)=-\frac{\partial}{\partial t}\vec{\nabla}\times\mathbf{B}=-\frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}+\mu_0\frac{\partial\mathbf{P}}{\partial t}\right)$$

Use the vector ID:

$$\mathbf{A}\times\left(\mathbf{B}\times\mathbf{C}\right)=\mathbf{B}\left(\mathbf{A}\cdot\mathbf{C}\right)-\mathbf{C}\left(\mathbf{A}\cdot\mathbf{B}\right)$$

$$\vec{\nabla}\times\left(\vec{\nabla}\times\mathbf{E}\right)=\vec{\nabla}\left(\vec{\nabla}\cdot\mathbf{E}\right)-\left(\vec{\nabla}\cdot\vec{\nabla}\right)\mathbf{E}=-\vec{\nabla}^2\mathbf{E}$$

$$\boxed{\vec{\nabla}^2\mathbf{E}-\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}=\mu_0\frac{\partial^2\mathbf{P}}{\partial t^2}}$$

"Inhomogeneous Wave Equation"

Maxwell's Equations in a Medium

- The induced polarization, **P**, contains the effect of the medium:

$$\vec{\nabla}^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2}$$

- Sinusoidal waves of all frequencies are solutions to the wave equation
- The polarization (**P**) can be thought of as the driving term for the solution to this equation, so the polarization determines which frequencies will occur.
- For linear response, **P** will oscillate at the same frequency as the input.

$$\mathbf{P}(\mathbf{E}) = \varepsilon_0 \chi \mathbf{E}$$

- In nonlinear optics, the induced polarization is more complicated:

$$\mathbf{P}(\mathbf{E}) = \varepsilon_0 \left(\chi^{(1)}\mathbf{E} + \chi^{(2)}\mathbf{E}^2 + \chi^{(3)}\mathbf{E}^3 + \ldots \right)$$

- The nonlinear terms lead to new frequencies and phase modulation.

Linear propagation of quasi-monochromatic fields

- Earlier we had worked with single-frequency fields, for example:

$$\mathbf{E}(z,t) = \hat{\mathbf{x}} E_x \cos(k_z z - \omega t)$$

- Now we want to work with field with a more general temporal shape.
  - Assume linear polarization, plane waves in z-direction
- For now, look at only the linear part of *P* :

$$\frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial^2 P_L}{\partial t^2} \qquad D = \varepsilon_0 E + P_L$$

- Group linear terms together

$$\rightarrow \frac{\partial^2 E}{\partial z^2} - \frac{1}{\varepsilon_0 c^2}\frac{\partial^2 D}{\partial t^2} = 0 \qquad \frac{1}{\varepsilon_0 \mu_0} = c^2$$

Wave equation in frequency space

- Represent all signals in ω space:

$$E(z,t) = \frac{1}{2\pi}\int E(z,\omega)e^{-i\omega t}\,d\omega$$

$$D(z,t) = \frac{1}{2\pi}\int D(z,\omega)e^{-i\omega t}\,d\omega$$

- Now we can connect *D* and *E*: $D(z,\omega) = \varepsilon_0 \varepsilon(\omega) E(z,\omega)$

- Put these expressions into the WE, do time derivatives inside integral: $\frac{\partial^2}{\partial t^2}E(z,t) = \frac{1}{2\pi}\int E(z,\omega)\left(\frac{\partial^2}{\partial t^2}e^{-i\omega t}\right)d\omega$

$$\frac{\partial^2}{\partial z^2}E(z,\omega) + \varepsilon(\omega)\frac{\omega^2}{c^2}E(z,\omega) = 0 \qquad k^2(\omega) = \varepsilon(\omega)\frac{\omega^2}{c^2}$$

- Now work to get back into time domain.

Field with slowly varying envelope

- We went to ω space to be able to easily include dispersion

$$\frac{\partial^2}{\partial z^2}E(z,\omega)+k^2(\omega)E(z,\omega)=0$$

- Represent field in terms of a slowly-varying amplitude

$$E(z,t)=A(z,t)\left(e^{i(k_0 z-\omega_0 t)}+c.c.\right) \qquad A(z,t)=\frac{1}{2\pi}\int A(z,\omega)e^{-i\omega t}\,d\omega$$

  – By shift theorem:

$$E(z,\omega)=A(z,\omega-\omega_0)e^{ik_0 z}$$

- Put this into the wave equation:

$$\frac{\partial^2}{\partial z^2}\left(A(z,\omega-\omega_0)e^{ik_0 z}\right)+k^2(\omega)A(z,\omega-\omega_0)e^{ik_0 z}=\left(\frac{\partial^2 A}{\partial z^2}+2ik_0\frac{\partial A}{\partial z}-k_0^2+k^2 A\right)e^{ik_0 z}$$

$$\frac{\partial^2 A}{\partial z^2}+2ik_0\frac{\partial A}{\partial z}+\left(k^2-k_0^2\right)A=0$$

Taylor expansion of dispersion

- Do a Taylor expansion for $k(\omega)$:

$$k(\omega) = k_0 + (\omega - \omega_0) k_1 + D \qquad D = \sum_{n=2}^{\infty} \frac{1}{n!} (\omega - \omega_0)^n k_n$$

D includes all high-order dispersion

$$k^2(\omega) = {\color{red}{k_0}^2} + 2k_0 k_1 (\omega - \omega_0) + {k_1}^2 (\omega - \omega_0)^2 + 2k_0 D + 2k_1 (\omega - \omega_0) D {\color{blue}{} + D^2}$$

$\color{blue}{D^2}$ → small

- Insert this expansion into the ω-domain WE:

$$\frac{\partial^2 A}{\partial z^2} + 2ik_0 \frac{\partial A}{\partial z} + \left(k(\omega)^2 - {\color{red}k_0^2} \right) A = 0$$

  - Terms in red cancel,

$$\frac{\partial^2 A}{\partial z^2} + 2ik_0 \frac{\partial A}{\partial z} + \left(2k_0 k_1 (\omega - \omega_0) + {k_1}^2 (\omega - \omega_0)^2 + 2k_0 D + 2k_1 (\omega - \omega_0) D \right) A = 0$$

Transform back to time domain

$$\frac{\partial^2 A}{\partial z^2} + 2ik_0 \frac{\partial A}{\partial z} + \left(2k_0 k_1 (\omega - \omega_0) + k_1{}^2 (\omega - \omega_0)^2 + 2k_0 D + 2k_1 (\omega - \omega_0) D\right) A = 0$$

- Now inverse FT to go back to time domain
  - Multiply by $e^{-i(\omega-\omega_0)t}$, integrate
  - Note that $FT^{-1}\left\{\omega^n A(\omega)\right\} = \left(i\frac{\partial}{\partial t}\right)^n \tilde{A}(t)$ $\qquad \tilde{D} = \sum_{n=2}^{\infty} \frac{1}{n!} k_n \left(i\frac{\partial}{\partial t}\right)^n$

$$\frac{\partial^2 \tilde{A}}{\partial z^2} + 2ik_0 \frac{\partial \tilde{A}}{\partial z} + \left(2ik_0 k_1 \frac{\partial}{\partial t} - k_1{}^2 \frac{\partial^2}{\partial t^2} + 2k_0 \tilde{D} + 2ik_1 \tilde{D} \frac{\partial}{\partial t}\right) \tilde{A} = 0$$

- For now, ignore high-order dispersion

$$\left(\frac{\partial^2}{\partial z^2} + 2ik_0 \frac{\partial}{\partial z} + 2ik_0 k_1 \frac{\partial}{\partial t} - k_1{}^2 \frac{\partial^2}{\partial t^2}\right) \tilde{A} = 0$$

  - This can be simplified by changing to a coordinate system moving with the pulse at the group velocity

Moving reference frame

- Change to reference frame moving at the group velocity

$$\left(\frac{\partial^2}{\partial z^2}+2ik_0\left(\frac{\partial}{\partial z}+k_1\frac{\partial}{\partial t}\right)-k_1{}^2\frac{\partial^2}{\partial t^2}\right)\tilde{A}=0$$

  - Change coordinates:

$$z'=z \qquad \frac{\partial}{\partial z}=\frac{\partial z'}{\partial z}\frac{\partial}{\partial z'}+\frac{\partial \tau}{\partial z}\frac{\partial}{\partial \tau}=\frac{\partial}{\partial z'}-k_1\frac{\partial}{\partial \tau}$$

$$\tau=t-k_1 z \qquad \frac{\partial}{\partial t}=\frac{\partial}{\partial \tau} \qquad \frac{\partial^2}{\partial z^2}=\left(\frac{\partial}{\partial z'}-k_1\frac{\partial}{\partial \tau}\right)^2=\frac{\partial^2}{\partial z'^2}-2k_1\frac{\partial}{\partial z'}\frac{\partial}{\partial \tau}+k_1{}^2\frac{\partial^2}{\partial \tau^2}$$

$$\left(\frac{\partial^2}{\partial z'^2}-2k_1\frac{\partial}{\partial z'}\frac{\partial}{\partial \tau}+\cancel{k_1{}^2\frac{\partial^2}{\partial \tau^2}}+2ik_0\left(\frac{\partial}{\partial z'}-\cancel{k_1\frac{\partial}{\partial \tau}}+\cancel{k_1\frac{\partial}{\partial \tau}}\right)-\cancel{k_1{}^2\frac{\partial^2}{\partial \tau^2}}\right)\tilde{A}=0$$

$$\left(\frac{\partial^2}{\partial z'^2}-2k_1\frac{\partial}{\partial z'}\frac{\partial}{\partial \tau}+2ik_0\frac{\partial}{\partial z'}\right)\tilde{A}=0 \qquad \boxed{\rightarrow\left(\frac{\partial^2}{\partial z'^2}+2ik_0\frac{\partial}{\partial z'}\left(1+i\frac{k_1}{k_0}\frac{\partial}{\partial \tau}\right)\right)\tilde{A}=0}$$

Simpler equation for envelope.

Slowly-varying envelope approx: SVEA

- So far, we haven't made any approximation about the duration of the pulse (or its bandwidth)
  - Assuming a carrier frequency doesn't itself introduce approximations
- Compare magnitude of components of equation:
  - In general, the envelope $A(z,t)$ will evolve over some length scale L (e.g. b/c of GVD): $\partial/\partial z' \sim 1/L$

$$\left(\frac{\partial^2}{\partial z'^2} + 2ik_0 \frac{\partial}{\partial z'}\left(1 + i\frac{k_1}{k_0}\frac{\partial}{\partial \tau}\right)\right)\tilde{A} = 0 \qquad \frac{\partial^2}{\partial z'^2} \sim \frac{1}{L^2} \qquad 2k_0\frac{\partial}{\partial z'} \sim \frac{4\pi}{\lambda_0 L}$$

  - So if $L \gg \dfrac{\lambda_0}{4\pi}$ we can ignore second derivative term

SVEA $\dfrac{\partial^2}{\partial z'^2} \to 0$

$$2ik_0 \frac{\partial}{\partial z'}\left(1 + i\frac{k_1}{k_0}\frac{\partial}{\partial \tau}\right)\tilde{A} = 0$$

  - Dropping this eliminates any counter-propagating solution: no back-reflections included in this approximation.

SVEA again

- We still have an extra time derivative

$$2ik_0 \frac{\partial}{\partial z'}\left(1 + i\frac{k_1}{k_0}\frac{\partial}{\partial \tau}\right)\tilde{A} = 0$$

  – Look at ratio:

  – $v_g \sim v_{ph}$ in order of magnitude

$$\frac{k_1}{k_0} = \frac{dk/d\omega|_{\omega_0}}{n\omega_0/c} = \frac{1}{\omega_0}\frac{v_{ph}}{v_g} \approx \frac{1}{\omega_0}$$

- Timescale for change τ_p $\qquad \partial/\partial\tau \sim 1/\tau_p$
  - If $\omega_0\tau_p$>>1, we can drop the time derivative.

$$\omega_0\tau_p \approx 2\frac{\omega_0}{\Delta\omega}$$

  - This approximation requires small fractional bandwidth.

$$\rightarrow 2ik_0 \frac{\partial}{\partial z'}\tilde{A} = 0$$

  - All this says is that the pulse shape doesn't change, but we assumed there was no high-order dispersion.

Dispersive propagation in the time domain

- Before changing to the moving coordinate system, we had

$$\left(\frac{\partial^2}{\partial z^2}+2ik_0\frac{\partial}{\partial z}+2ik_0k_1\frac{\partial}{\partial t}-k_1^{\,2}\frac{\partial^2}{\partial t^2}+2k_0\tilde{D}+2ik_1\tilde{D}\frac{\partial}{\partial t}\right)\tilde{A}=0 \qquad \tilde{D}=\sum_{n=2}^{\infty}\frac{1}{n!}k_n\left(i\frac{\partial}{\partial t}\right)^n$$

  - In moving ref frame, and with SVEA, this is now:

$$\left(2ik_0\frac{\partial}{\partial z'}+2k_0\tilde{D}+2ik_1\tilde{D}\frac{\partial}{\partial \tau}\right)\tilde{A}=0\rightarrow\left(2ik_0\frac{\partial}{\partial z'}+2k_0\tilde{D}\left(1+i\frac{k_1}{k_0}\frac{\partial}{\partial \tau}\right)\right)\tilde{A}=0$$

  - Term in blue is small as in previous slide, so dispersive propagation follows the equation:

$$\left(2ik_0\frac{\partial}{\partial z'}+2k_0\tilde{D}\right)\tilde{A}=0$$

  - For second-order dispersion only,

$$\tilde{D}=\sum_{n=2}^{\infty}\frac{1}{n!}k_n\left(i\frac{\partial}{\partial t}\right)^n\rightarrow\frac{1}{2!}k_2\left(i\frac{\partial}{\partial t}\right)^2=-\frac{1}{2}k_2\frac{\partial^2}{\partial t^2}$$

$$\boxed{\frac{\partial \tilde{A}}{\partial z'}=-i\frac{1}{2}k_2\frac{\partial^2\tilde{A}}{\partial t^2}}$$

Nonlinear propagation

- Polarization has a nonlinear component

$$\frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial^2 P_L}{\partial t^2} + \mu_0 \frac{\partial^2 P_{NL}}{\partial t^2}$$

- Treat $\mu_0 \frac{\partial^2 P_{NL}}{\partial t^2}$ as a source term in all previous eqns.

$$\tilde{P}_{NL}(z,t) = 3\varepsilon_0 \chi^{(3)} \left|\tilde{A}(z,t)\right|^2 \tilde{A}(z,t) e^{i(k_0 z - \omega_0 t)} \qquad n_2 I = \frac{3\chi^{(3)}}{2n_0}\left|\tilde{A}\right|^2$$

- Working with the carrier and envelope:

$$\tilde{P}_{NL}(z,t) = \tilde{p}(z,t) e^{i(k_0 z - \omega_0 t)}$$

$$\frac{\partial \tilde{P}_{NL}}{\partial t} = \left(-i\omega_0 \tilde{p} + \frac{\partial \tilde{p}}{\partial t}\right) e^{i(k_0 z - \omega_0 t)} = -i\omega_0 \left(1 + \frac{i}{\omega_0}\frac{\partial}{\partial t}\right) \tilde{p}\, e^{i(k_0 z - \omega_0 t)}$$

$$\rightarrow \frac{\partial^2 \tilde{P}_{NL}}{\partial t^2} = -\omega_0{}^2 \left(1 + \frac{i}{\omega_0}\frac{\partial}{\partial t}\right)^2 \tilde{p}\, e^{i(k_0 z - \omega_0 t)} \approx -\omega_0{}^2 \tilde{p}\, e^{i(k_0 z - \omega_0 t)}$$

Drop red term by SVEA

Nonlinear Schrodinger Equation (NLS)

$$\mu_0 \frac{\partial^2 \tilde{P}_{NL}}{\partial t^2} \approx -\mu_0 \omega_0{}^2 \tilde{p}\, e^{i(k_0 z - \omega_0 t)} = -3\mu_0 \varepsilon_0 \omega_0{}^2 \chi^{(3)} \left|\tilde{A}\right|^2 \tilde{A} e^{i(k_0 z - \omega_0 t)}$$

$$= -3\chi^{(3)} \frac{\omega_0{}^2}{c^2} \left|\tilde{A}\right|^2 \tilde{A} e^{i(k_0 z - \omega_0 t)}$$

- Add NL contribution to RHS:

$$\left(2ik_0 \frac{\partial}{\partial z'} + 2k_0 \tilde{D} \right) \tilde{A} = -3\chi^{(3)} \frac{\omega_0{}^2}{c^2} \left|\tilde{A}\right|^2 \tilde{A}$$

$$\left(i \frac{\partial}{\partial z'} + \tilde{D} \right) \tilde{A} = -\frac{\omega_0}{c} n_2 I \tilde{A}$$

- With only 2nd order term in dispersion:

$$\boxed{\frac{\partial \tilde{A}}{\partial z'} = -i \frac{1}{2} k_2 \frac{\partial^2 \tilde{A}}{\partial t^2} + i \frac{\omega_0}{c} n_2 I \tilde{A}}$$

Operator form

$$\partial_z A_0 = \left[\hat{D} + \hat{N} \right] A_0$$

Few-Cycle Pulses by External Compression

Sandro De Silvestri, Mauro Nisoli, Giuseppe Sansone, Salvatore Stagira, and Orazio Svelto

F.X. Kärtner (Ed.): Few-Cycle Laser Pulse Generation and Its Applications, Topics Appl. Phys. **95**, 137–178 (2004)

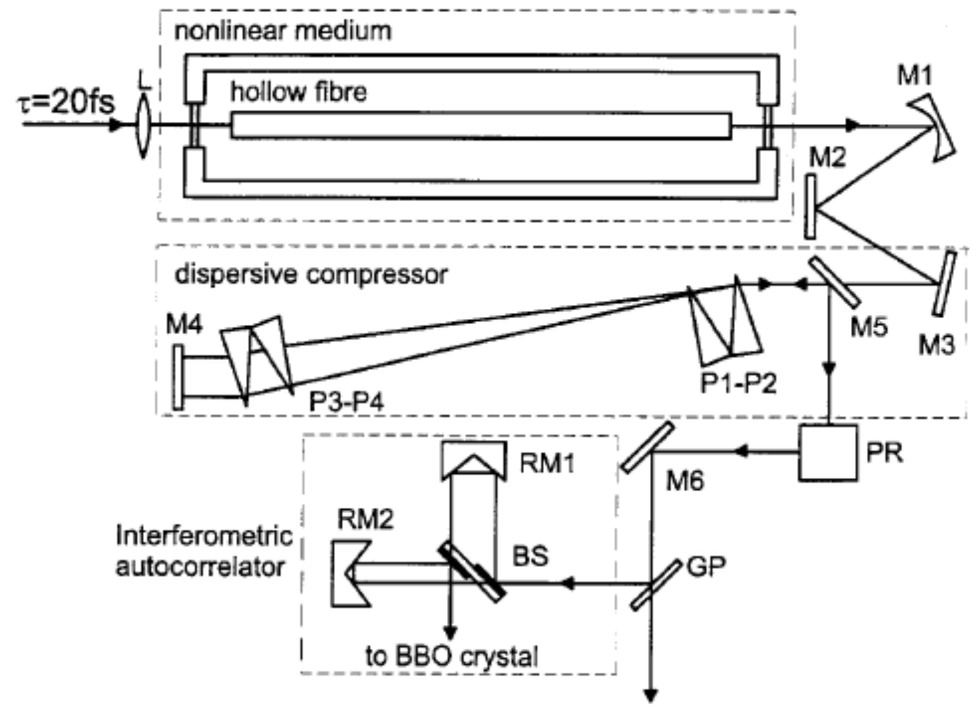
$$\Delta\beta = \frac{(\omega_0/c)\iint \Delta n |F(x,y)|^2 \,\mathrm{d}x\,\mathrm{d}y}{\iint |F(x,y)|^2 \,\mathrm{d}x\,\mathrm{d}y}\,.$$

Change in propagation constant is averaged over the mode. SPM applies to whole mode.

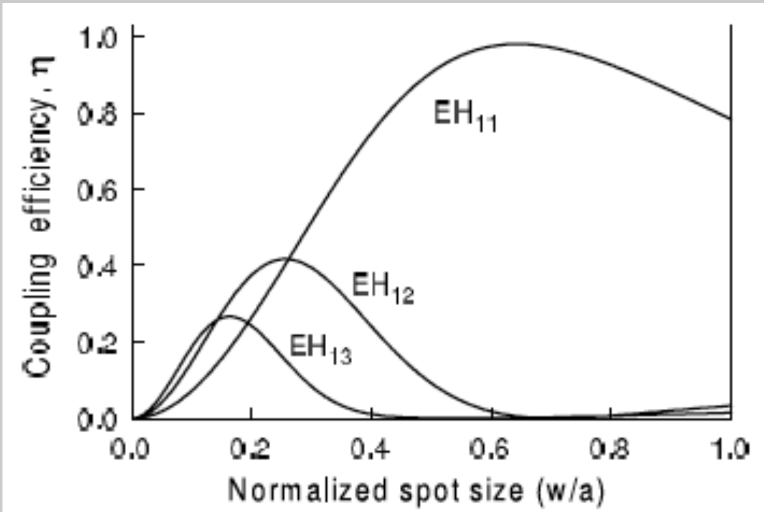
$$\Delta n \propto n_2 \left|F(x,y)\right|^2$$

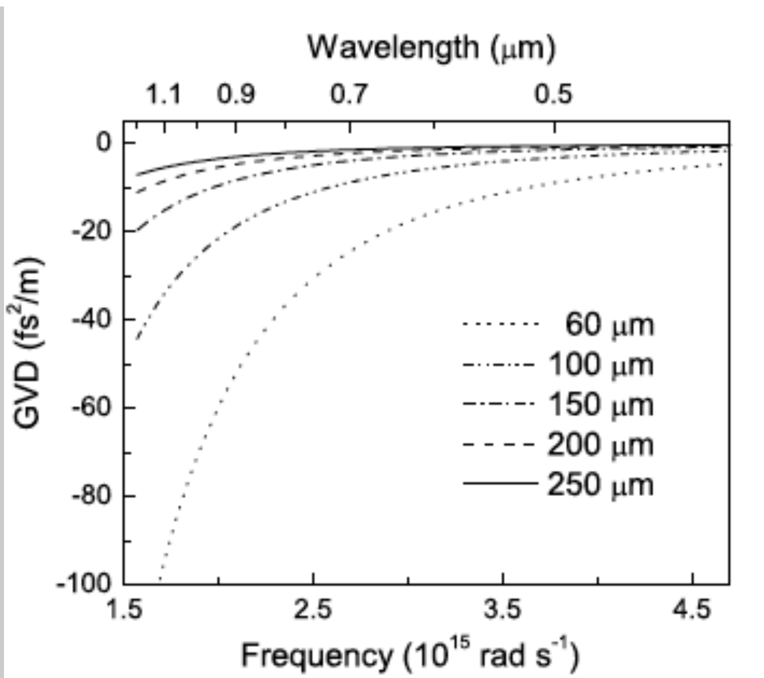
$$\beta(\omega) = \frac{\omega n_{\mathrm{core}}(\omega)}{c}\left[1 - \frac{1}{2}\left(\frac{u_m c}{\omega n_{\mathrm{core}}(\omega) a}\right)^2\right]$$

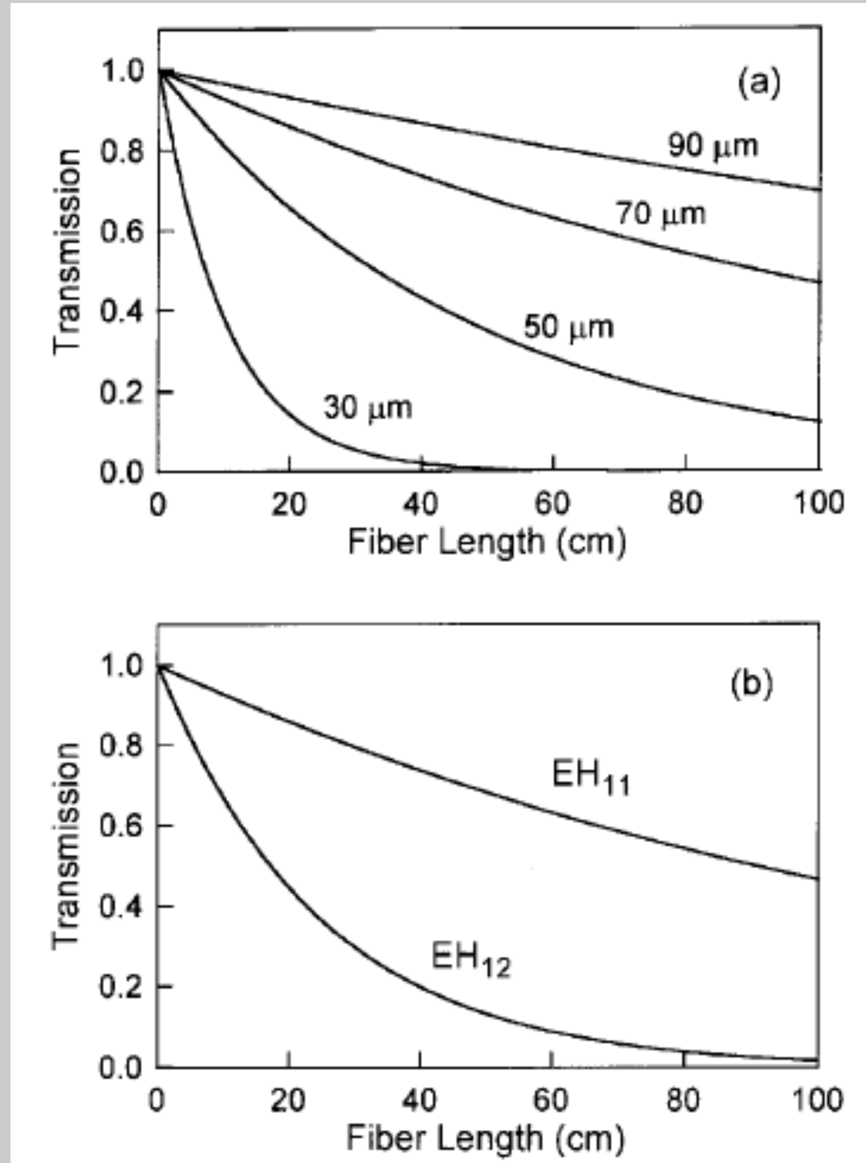
$$+\frac{\mathrm{i}}{a^3}\left(\frac{u_m c}{\omega n_{\mathrm{core}}(\omega)}\right)^2 \frac{n^2(\omega)+1}{\sqrt{n^2(\omega)-1}}$$



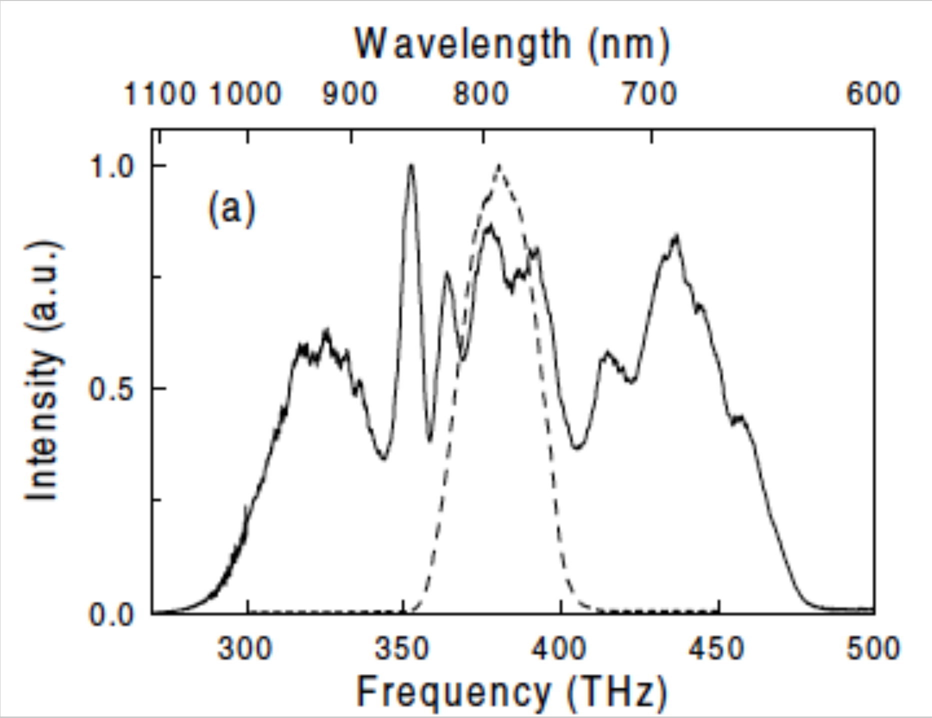
Properties of hollow-core waveguides

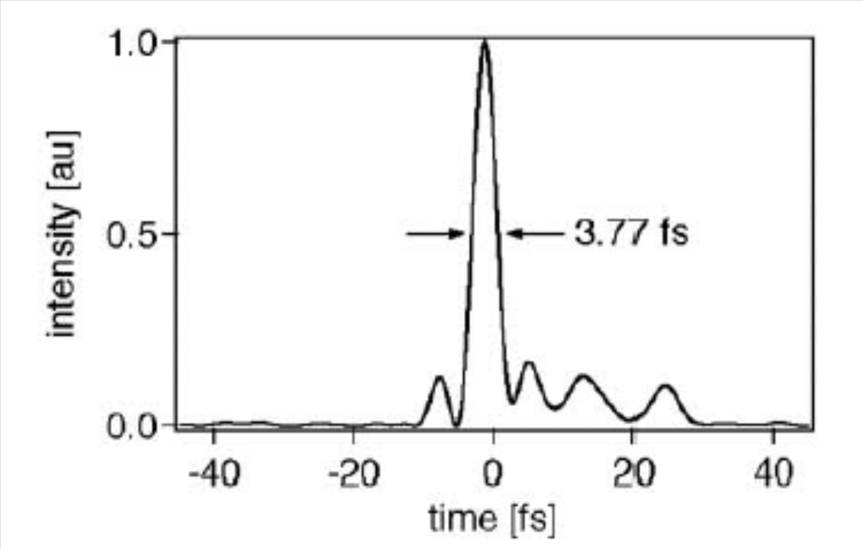






Output spectrum and pulse shape

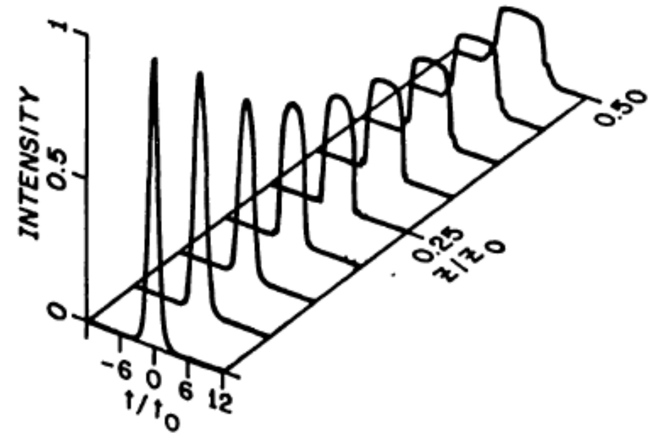




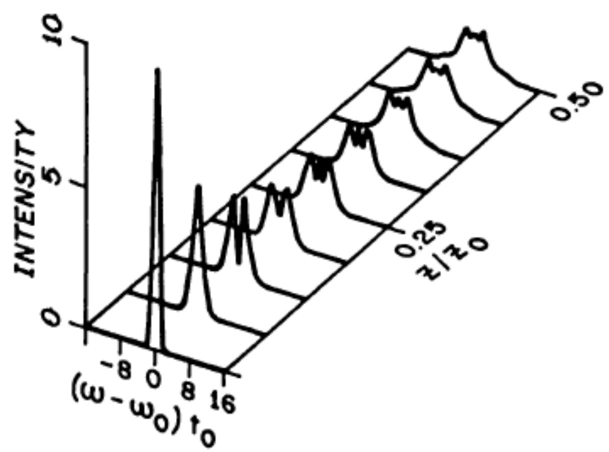
Compression of optical pulses chirped by self-phase modulation in fibers

W. J. Tomlinson,* R. H. Stolen, and C. V. Shank
AT&T Bell Laboratories, Holmdel, New Jersey 07733

Vol. 1, No. 2/April 1984/J. Opt. Soc. Am. B 139



(a)



(b)