MATH 348 - Advanced Engineering Mathematics June 25, 2009

Homework 4, Summer 2009 **Due**: June 30 , 2009

Eigenvalues - Eigenvectors - Diagionalization - Spectral Decomposition - Applications

1. Find the eigenvalues and eigenvectors of the following matrix.

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

2. Given,

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}.$$

Determine the eigenvalues and eigenfunctions associated with the system of differential equations $\frac{d\mathbf{x}}{dt} = \mathbf{A} \cdot \mathbf{x}(t)$.

3. Given,

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$$

If $\mathbf{A}$ is diagonalizable, then determine $\mathbf{D}$ and $\mathbf{P}$ associated with its decomposition $\mathbf{PDP}^{-1}$. <u>Do not find $\mathbf{P}^{-1}$.</u>

4. Square matrices having columns whose entries sum to 1 are often called stochastic matrices. Those with only non-negative entries, for some power, are called *regular* stochastic matrices. Given a random process, with an initial state $\mathbf{x}_0$, the application of $\mathbf{P}$ on $\mathbf{x}_0$ discretely steps the process forward in time. That is $\mathbf{x}_{n+1} = \mathbf{P}\mathbf{x}_n = \mathbf{P}^n\mathbf{x}_0$, $n = 1, 2, 3, \ldots$. If a matrix is a *regular* stochastic matrix then there exists a steady-state vector $\mathbf{q}$ such that $\mathbf{Pq}$=$\mathbf{q}$. This vector determines the long term probabilities associated with an arbitrary inital state $\mathbf{x}_0$. The sequence of states, $\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n+1}\}$, is called a *Markov Chain.* Given the regular stochastic matrix:

$$\mathbf{P} = \begin{bmatrix} .1 & .6 \\ .9 & .4 \end{bmatrix}.$$

   (a) Show that the steady-state vector of $\mathbf{P}$ is $\mathbf{q} = \begin{bmatrix} \frac{2}{5} & \frac{3}{5} \end{bmatrix}^{\mathrm{T}}$.

   (b) Find the matrices $\mathbf{D}$ and $\mathbf{Q}$ such that $\mathbf{P}$= $\mathbf{QDQ}^{-1}$. That is, diagonalize the matrix $\mathbf{P}$.

   (c) Show that $\lim_{n\to\infty} \mathbf{P}^n\mathbf{x}_0 = \mathbf{q}$ where $\mathbf{x}_0 = [x_1, \, x_2]^{\mathrm{T}}$ is an arbitrary vector in $\mathbb{R}^2$ such that $x_1 + x_2 = 1$.

5. Recall the Pauli Spin Matrix from homework 1,

$$\sigma_2 = \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

   (a) Show that σ_y is self-adjoint. [1]

   (b) Find the orthogonal diagonalization of σ_y.[2]

   (c) Show that $\sigma_y = \lambda_1 \mathbf{x}_1 \mathbf{x}_1^{\mathrm{H}} + \lambda_2 \mathbf{x}_2 \mathbf{x}_2^{\mathrm{H}}$, where $\mathbf{x}_1$ and $\mathbf{x}_2$ are the normalized eigenvectors from part (b).

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[1] Recall that self-adjoint means that $\mathbf{A} = \mathbf{A}^{\mathrm{H}} = \bar{\mathbf{A}}^{\mathrm{T}}$. See the linear algebra handout on `ticc.mines.edu` for the definitions.

[2] This will require you to normalize vectors that contain imaginary numbers. Generally to normalize a vector $\mathbf{x}$ we make a new vector $\hat{\mathbf{x}} = \frac{\mathbf{x}}{|\mathbf{x}|} = \frac{\mathbf{x}}{\sqrt{\mathbf{x}^{\mathrm{T}}\mathbf{x}}}$. However, when using vectors with imaginary entries we must use the adjoint in our definition of inner-product. That is, we take a normalized vector to be, $\hat{\mathbf{x}} = \frac{\mathbf{x}}{|\mathbf{x}|} = \frac{\mathbf{x}}{\sqrt{\mathbf{x}^{\mathrm{H}}\mathbf{x}}}$. If you do not do this then then your inner-products will become zero and your normalized vectors will be undefined, which I guess they should be in real-space.