Exam 2

Note Title 12/1/2006

$$f(t) = \begin{cases} 0 & t < 0 \\ e^{-\mu t} & t > 0 \end{cases} \qquad \mu > 0$$

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt$$

$$= \frac{1}{2\pi}\int_{0}^{\infty} e^{-\mu t}\, e^{i\omega t}\, dt$$

$$= \frac{1}{2\pi}\int_{0}^{\infty} e^{(i\omega - \mu)t}\, dt$$

$$= \frac{1}{2\pi}\, \frac{1}{i\omega - \mu}\, e^{(i\omega-\mu)t}\Big|_{0}^{\infty}$$

$$\left| e^{-\mu t} e^{i\omega t} \right| \leq \left| e^{-\mu t} \right|$$

so

$$\frac{1}{2\pi} \frac{1}{i\omega - \mu}$$

$$\frac{1}{i\omega - \mu} \frac{-i\omega - \mu}{-i\omega - \mu} = - \frac{\mu + i\omega}{\omega^2 + \mu^2}$$

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2) $$\frac{d}{dx}\left((1-x^2) \frac{dT}{dx} \right) - \mu T(x) = 0$$

$$(1 - x^2) T'' - 2xT' - \mu T = 0$$

Let $T(x) = \sum_{N=0}^{\infty} a_N x^N$

$$T' = \sum a_N N x^{N-1}$$

$$x T' = \sum_{N=0}^{\infty} a_N N x^N$$

$$T'' = \sum a_N (N-1) N x^{N-2}$$

$$-x^2 T'' = \sum_{N=0}^{\infty} a_N (N-1) N x^N$$

$$T'' = \sum_{N=0}^{\infty} a_{N+2} (N+1)(N+2) x^N$$

🚩 $$\Rightarrow \sum_{n=0}^{\infty} \Big[a_{n+2}(n+1)(n+2) - a_n(n-1)n - 2a_n n - \mu a_n\Big] x^n = 0$$

$$\Rightarrow a_{n+2}(n+1)(n+2) - \big[(n-1)n + 2n + \mu\big] a_n = 0$$

$$\boxed{a_{n+2} = -\frac{(n-1)n + 2n + \mu}{(n+1)(n+2)} a_n}$$

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3) $\nabla^2 V(r) = 0$ $\quad V(r_0) = 0$

$V(r_1) = V_0$

since $V = V(r)$ only $\nabla^2 V$

$$= \boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) = 0}$$

$$\Rightarrow \frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) = 0$$

$$\Rightarrow r^2 \frac{\partial \psi}{\partial r} = \text{constant}$$

$$\Rightarrow \frac{\partial \psi}{\partial r} = \frac{c}{r^2}$$

$$\Rightarrow \quad \boxed{\psi(r) = a - \frac{c}{r}}$$

$$\psi(r_0) = 0 \Rightarrow a - \frac{c}{r_0} = 0$$

$$\psi(r_1) = V_0 \Rightarrow a - \frac{c}{r_1} = V$$

$$a = \frac{c}{r_0}$$

$$\frac{c}{r_0} - \frac{c}{r_1} = V_0 \Rightarrow c = \frac{V_0}{\frac{1}{r_0} - \frac{1}{r_1}}$$

$$= \frac{r_0 r_1}{r_1 - r_0} V_0$$

$$\psi(r) = \frac{c}{r_0} - \frac{r_0 r_1}{r_1 - r_0} V_0$$

$$= \frac{r_1}{r_1 - r_0} V_0 - \frac{r_0 r_1}{r_1 - r_0} V_0 \frac{1}{r}$$

or $$\boxed{\psi(r) = \frac{r_1 V_0}{r_1 - r_0}\left(1 - \frac{r_0}{r}\right)}$$

check $\psi(r_0) = 0$

$$\psi(r_1) = \frac{r_1 v_0}{r_1 - r_0}\left(1 - \frac{r_0}{r_1}\right)$$

$$= \frac{r_1 - r_0}{r_1 - r_0} v_0 = v_0 \quad \checkmark$$

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4) $$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$$

$$u(x,t) = X(x)\,T(t)$$

$$X'' T - \frac{1}{c^2} X \ddot{T} = 0$$

$$\frac{X''}{X} - \frac{1}{c^2}\frac{\ddot{T}}{T} = 0$$

$$\frac{X''}{X} = \frac{1}{c^2}\frac{\ddot{T}}{T} = -k^2$$

$$\ddot{T} + \omega^2 T = 0 \qquad ck = \omega$$

$$X'' + k^2 X = 0$$

$$X(x) = A \sin kx + B \cos kx$$

<u>BC</u> $\quad X'(0) = X'(L) = 0$

$$X' = kA \cos kx - kB \sin kx$$

$$X'(0) = kA = 0 \Rightarrow A = 0$$

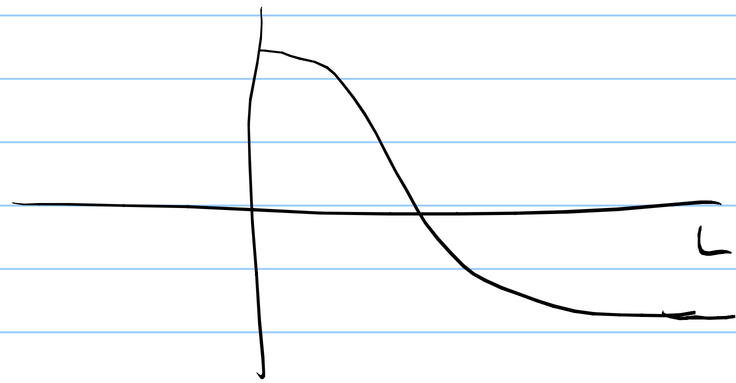
$$X'(L) = -kB \sin(kL) = 0$$

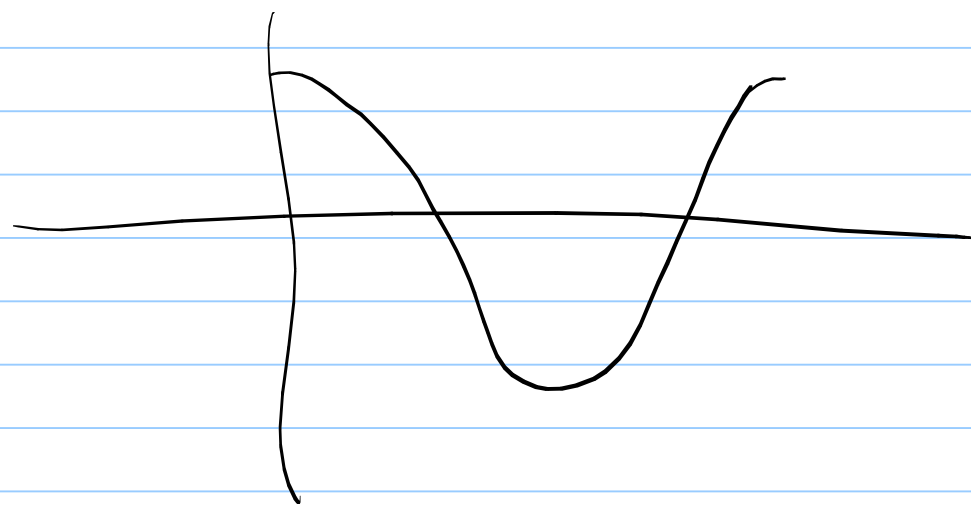
$$\Rightarrow kL = N\pi$$

So $\boxed{X_N(x) = B \cos(k_N x) \quad \text{where} \quad k_N = \frac{N\pi}{L}}$

$$ck_N = \omega_N \Rightarrow \boxed{\omega_N = \frac{N\pi c}{L}}$$

modes : $\cos\left(\frac{n\pi x}{L}\right)$





slope of string is zero