Optional

We can compute the rate of change of the unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ from Equations 3-45 and 3-46 and use them to compute the velocity and acceleration vectors for circular motion. We have

$$\frac{d\hat{\mathbf{r}}}{dt} = \frac{d}{dt}(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})$$

$$= -\sin\theta\frac{d\theta}{dt}\mathbf{i} + \cos\theta\frac{d\theta}{dt}\mathbf{j}$$

$$= \frac{d\theta}{dt}(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}) = \frac{d\theta}{dt}\hat{\boldsymbol{\theta}} \qquad \text{3-47}$$

$$\frac{d\hat{\boldsymbol{\theta}}}{dt} = \frac{d}{dt}(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j})$$

$$= -\cos\theta\frac{d\theta}{dt}\mathbf{i} - \sin\theta\frac{d\theta}{dt}\mathbf{j}$$

$$= -\frac{d\theta}{dt}(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}) = \frac{-d\theta}{dt}\hat{\mathbf{r}} \qquad \text{3-48}$$

Let us now find the velocity and acceleration vectors by direct differentiation of the position vector for a particle moving in a circle

$$\mathbf{r} = r\hat{\mathbf{r}} \qquad \mathbf{v} = \frac{d\mathbf{r}}{dt} = r\frac{d\hat{\mathbf{r}}}{dt} = r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}$$

Since

$$r\frac{d\theta}{dt} = \frac{ds}{dt} = v$$

the speed, we have

$$\mathbf{v} = v\hat{\boldsymbol{\theta}}$$

Then the acceleration is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = v\frac{d\hat{\boldsymbol{\theta}}}{dt} + \frac{dv}{dt}\hat{\boldsymbol{\theta}}$$

where we have used the product rule for differentiating $\mathbf{v}$ when both quantities may depend on time. Using Equation 3-48 for $d\hat{\boldsymbol{\theta}}/dt$, we obtain

$$\mathbf{a} = v\frac{-d\theta}{dt}\hat{\mathbf{r}} + \frac{dv}{dt}\hat{\boldsymbol{\theta}}$$

This can be written in the usual form if we note that $d\theta/dt = v/r$:

$$\mathbf{a} = -\frac{v^2}{r}\hat{\mathbf{r}} + \frac{dv}{dt}\hat{\boldsymbol{\theta}}$$

Review

A Define, explain, or otherwise identify:

Displacement, 49
Vector, 50
Scalar, 50
Component of a vector, 50
Unit vector, 52
Vector equality, 53
Radius vector, 56
Position vector, 56
Range, 61
Centripetal acceleration, 66
Tangential acceleration, 69

B True or false:

1. The instantaneous velocity vector is always in the direction of motion.

2. The instantaneous acceleration vector is always in the direction of motion.